Noncommutative $SO(2,3)$ model

Voja Radovanović

University of Belgrade, Faculty of Physics,
Belgrade, Serbia

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Overview

Introduction

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GR and QFT are two pillars of modern physics, but both theories suffer from singularities. QFT and GR encounter problems small distances. There is no consistent (i. e. renormalizable and unitary) quantum theory of gravity. Modifications of QFT and GR are needed; point-particles or/and space-time structure.
Many attempts: String theory, Loop Quantum Gravity,..
One possibility is noncommutativity among space time coordinates. It is given by

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x) \ .$$

Canonical noncommutativity

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} = \mathrm{const.}$$

Different models are constructed on canonical NC spacetime: ϕ^4,QED, standard model, SUSY models;
renormalizability, unitarity, phenomenological consequences, ...

Noncommutative Gravity

GR is based on diffeomorphism symmetry. It is difficult to generalize this symmetry to NC space-time.

Many attempts:

- **NC spectral geometry** [Chamseddine, Connes, Marcolli '07; Chamseddine, Connes, Mukhanov '14].
- **Emergent gravity** [Steinacker '10, '16]
- **Frame formalism, operator description** [Burić, Madore '14; Fritz, Majid '16].
- **Twist approach** [Wess et al. '05, '06; Ohl, Schenckel '09; Castellani, Aschieri '09; Aschieri, Schenkel '14].
- **NC gravity as a gauge theory of Lorentz/Poincaré group** [Chamseddine '01,'04, Cardela, Zanon '03, Aschieri, Castellani '09,'12; Dobrski '16].
- **Nonassociative gravity** [R. Blumenhagen, M. Fuchs, '16; P. Aschieri, M. Dimitrijevic and R. Szabo, '18

Commutative $SO(2,3)$ model

Consider a gauge theory with $SO(2,3)$ as a gauge group.
$SO(2,3)$ is the isometry group anti de Sitter space.
Anti de Sitter space is a maximally symmetric space with a negative constant curvature.
M_{AB}-generators of $SO(2,3)$ group
$A, B, \dots$ take values $0, 1, 2, 3, 5$.
Commutation relations:

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}), \quad (1)$$

$\eta_{AB} = \mathrm{diag}(+, -, -, -, +)$ is $5D$ metric.

Clifford generators Γ_A in $5D$ Minkowski space satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \ . \tag{2}$$

M_{AB} are

$$M_{AB} = \frac{i}{2}[\Gamma_A, \Gamma_B] \ . \tag{3}$$

$\gamma_a, \ (a = 0, 1, 2, 3)$ are the gamma matrices in $4D$ Minkovski spacetime

The gamma matrices in $5D$ are

$$\Gamma_A = (i\gamma_a\gamma_5, \gamma_5) \ .$$

γ_5 is defined by $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

It is easy to show that

$$\begin{aligned} M_{ab} &= \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab} \ , \\ M_{5a} &= \frac{i}{2}\gamma_a \ . \end{aligned} \tag{4}$$

If we introduce momenta $P_a = \frac{1}{l} M_{a5}$,where l is a constant with dimensions of length AdS algebra (1) becomes

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}) \\ [M_{ab}, P_c] &= i(\eta_{bc} P_a - \eta_{ac} P_b) \\ [P_a, P_b] &= -i\frac{1}{l^2} M_{ab} \ . \end{aligned} \tag{5}$$

In the limit $l \to \infty$ AdS algebra reduces usual Poincare algebra in $4D$ spacetime. (Wigner-Inonu contraction)

$SO(2,3)$ gauge potential:

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB} M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma^{ab} - \frac{1}{2}\omega_\mu^{a5}\gamma_a \quad (6)$$

Transformation law

$$\delta_\epsilon \omega_\mu = \partial_\mu \epsilon + i[\epsilon, \omega_\mu] \quad (7)$$

Decomposition: ω_μ^{AB} *to* ω_μ^{ab}, ω_μ^{a5},
ω_μ^{ab} is a spin connection
$\omega_\mu^{a5} = \frac{1}{l} e_\mu^a$ are vierbeins (tetrads).

The field strength

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2} F^{AB}_{\mu\nu} M_{AB}$$
$$= \frac{1}{2} F^{ab}_{\mu\nu} M_{ab} + F^{a5}_{\mu\nu} M_{a5} \ , \qquad (8)$$

where

$$F^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} - \frac{1}{l^2}(e^a_\mu e^b_\nu - e^b_\mu e^a_\nu) \ . \qquad (9)$$

Reiman curvature tensor is

$$R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^{cb}_\nu - \omega^{bc}_\mu \omega^{ca}_\nu \qquad (10)$$

Torsion

$$l F^{a5}_{\mu\nu} = \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu = T^a_{\mu\nu} \qquad (11)$$

We introduce an auxiliary field $\phi = \phi^A \Gamma_A$.
Transformation law:

$$\delta_\epsilon \phi = i[\epsilon, \phi] \tag{12}$$

This field satisfies a constraint $\phi_A \phi^A = l^2$.
Action:

$$\begin{aligned} S_1 &= \frac{il}{64\pi G_N} \int \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left(F_{\mu\nu} F_{\rho\sigma} \phi\right) \\ S_2 &= \frac{1}{128\pi G_N l} \mathrm{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} D_\rho \phi D_\sigma \phi \phi + c.c, && (13) \\ S_3 &= -\frac{i}{128\pi G_N l} \mathrm{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} D_\mu \phi D_\nu \phi D_\rho \phi D_\sigma \phi \phi && (14) \end{aligned}$$

S_1 is Stelle-West action (1980).

$$S = c_1 S_1 + c_2 S_2 + c_3 S_3 \tag{15}$$

is invariant under the $SO(2,3)$ gauge transformations.
We reduce the local anti de Sitter symmetry down to the local Lorentz symmetry:

$$SO(2,3) \to SO(1,3)$$

After symmetry breaking (i. e. $\phi^a = 0, \phi^5 = l$) these actions reduce to

$$\begin{aligned}
S_1 &= -\frac{1}{16\pi G_N} \int \mathrm{d}^4 x \Big(\frac{l^2}{16} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}^{\ \ ab} R_{\rho\sigma}^{\ \ cd} + \sqrt{-g} (R - \frac{6}{l^2}) \Big) , \\
S_2 &= -\frac{1}{16\pi G_N} \int d^4 x \sqrt{-g} \Big(R - \frac{12}{l^2} \Big), \\
S_3 &= -\frac{1}{16\pi G_N} \int d^4 x \sqrt{-g} \Big(-\frac{12}{l^2} \Big).
\end{aligned}$$

We define a general commutative model to be:

$$\begin{aligned} S &= c_1 S_1 + c_2 S_2 + c_3 S_3 \\ &= -\frac{1}{16\pi G_N}\int d^4x\Big(c_1 \frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{\ ab}R_{\rho\sigma}^{\ cd} \\ &\quad +\sqrt{-g}\big((c_1+c_2)R - \frac{6}{l^2}(c_1+2c_2+2c_3)\big)\Big), \qquad (16) \end{aligned}$$

with $e_\mu{}^a = \frac{1}{l}\omega_\mu^{a5}$, $\sqrt{-g} = \det e_\mu^a$, $R = R_{\mu\nu}^{\ ab} e_a{}^\mu e_b{}^\nu$. The constants c_1, c_2 and c_3 are arbitrary and can be determined from $c_1 + c_2 = 1$, and the cosmological constant is given by

$$\Lambda = -3\frac{1+c_2+2c_3}{l^2} \ .$$

Note that the cosmological constant Λ can be positive, negative or zero, regardless of the symmetry of our model. In this action the vielbeins and spin connection are independent variables. Varying the action with respect to the spin connection we obtain an equation which relates connection and vielbeins. The commutative action is invariant under the Lorentz gauge transformations by construction. In addition this action posses invariance under general coordinate transformations. This action will be our starting point for the construction of a noncommutative gravity theory.

References:

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Stelle, West, PRD, 1980

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F. Wilczek, PRL, 1998

Seiberg-Witten map

We work in the $\theta-$constant space or canonical NC space. The canonical NC can be introduced by replacing the usual product by the Moyal-Weyl $\star$ product

$$f(x) \star g(x) = e^{\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y\to x}\ , \qquad (17)$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix.
It is a small deformation parameter.

$$[x^\mu \overset{\star}{,} x^\nu] = i\theta^{\mu\nu}$$

The commutative quantities replace by their noncommutative counterparts.

$$
\begin{aligned}
\epsilon,\ \Phi,\ \Psi,\ \omega_\mu, &\quad\rightarrow\quad \hat{\Lambda}_\epsilon,\ \hat{\Phi},\ \hat{\Psi},\ \hat{\omega}_\mu, \\
F_{\mu\nu} &\quad\rightarrow\quad \hat{F}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \stackrel{\star}{,} \hat{\omega}_\nu] \\
\delta_\epsilon \Psi = i\epsilon\Psi &\quad\rightarrow\quad \delta^\star_\epsilon \hat{\Psi} = i\hat{\Lambda}_\epsilon \star \hat{\Psi} \\
\delta_\epsilon \Phi = i[\epsilon, \Phi] &\quad\rightarrow\quad \delta^\star_\epsilon \hat{\Phi} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\Phi}] \\
\delta_\epsilon \omega_\mu = \partial_\mu \epsilon + i[\epsilon, \omega_\mu] &\quad\rightarrow\quad \delta^\star_\epsilon \hat{\omega}_\mu = \partial_\mu \hat{\Lambda}_\epsilon + i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\omega}_\mu] \\
\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}] &\quad\rightarrow\quad \delta^\star_\epsilon \hat{F}_{\mu\nu} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{F}_{\mu\nu}]
\end{aligned}
$$

Commutative and noncommutative symmetries which correspond to the same gauge group can be related by the Seiberg-Witten map: the map enables one to express the noncommutative variables in terms of the commutative variables. In that way no new degrees of freedom are introduced. SW map can also be seen as an expansion in $\theta^{\mu\nu}$, so the SW approach is known as a $\theta-$expanded theory.

NC Gravity Action

The NC generalization of (16) is given by

$$S_{NC} = c_1 S_{1NC} + c_2 S_{2NC} + c_3 S_{3NC} \,, \tag{18}$$

with

$$\begin{aligned} S_{1NC} &= \frac{il}{64\pi G_N} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi} \,, \\ S_{2NC} &= \frac{1}{64\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} \hat{\phi} \star \hat{F}_{\mu\nu} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} + c.c. \,, \\ S_{3NC} &= -\frac{i}{128\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4 x \, \varepsilon^{\mu\nu\rho\sigma} D_\mu \hat{\phi} \star D_\nu \hat{\phi} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi} \,. \end{aligned}$$

It is invariant under the NC $SO(2,3)_\star$ gauge symmetry and the SW map guarantees that after the expansion it will be invariant under the commutative $SO(2,3)$ gauge symmetry.

Seiberg-Witten expansion

$$S_{NC} = S^{(0)} + S^{(1)} + S^{(2)} + \dots \ . \tag{19}$$

$$S_{NC}^{(1)} = 0$$

After the symmetry breaking the field $\phi^a = 0, \ \phi^5 = l$. We are interested in the low energy expansion we keep only the terms of the zeroth, the first and the second order in the derivatives of vierbeins (linear in $R_{\alpha\beta\gamma\delta}$, quadratic in $T^a_{\alpha\beta}$):

$$
\begin{aligned}
S_{NC} = \quad & \frac{1}{128\pi G_N l^4}\int \mathrm{d}^4x\, e\theta^{\alpha\beta}\theta^{\gamma\delta}\Big((-2+12c_2+38c_3)R_{\alpha\beta\gamma\delta} \\
& +(4-18c_2-44c_3)R_{\alpha\gamma\beta\delta}-(6+22c_2+36c_3)g_{\beta\delta}R_{\alpha\gamma}+\frac{6+28c_2+56c_3}{l^2}g_{\alpha\gamma}g_{\beta\delta} \\
& +(5-\frac{9}{2}c_2-7c_3)T^a_{\alpha\beta}T_{\gamma\delta a}+(-10+\frac{9}{2}c_2+14c_3)T^a_{\alpha\gamma}T_{\beta\delta a}+(3-3c_2-2c_3)T_{\alpha\beta\gamma}T_{\delta\mu}^{\ \ \mu} \\
& +(1+2c_2)T_{\alpha\beta\rho}T^{\rho}_{\ \gamma\delta}+8T_{\alpha\gamma\delta}T_{\beta\mu}^{\ \ \mu}-(2c_2+4c_3)T_{\alpha\gamma\rho}T^{\rho}_{\ \delta\beta} \\
& +(2c_2+4c_3)g_{\beta\delta}T_{\gamma\sigma}^{\ \ \sigma}T_{\alpha\rho}^{\ \ \rho}-(2c_2+4c_3)T_{\alpha\rho\sigma}T_{\gamma}^{\ \sigma\rho}g_{\beta\delta}+(-2+4c_2+18c_3)T_{\alpha\beta\gamma}e^{\rho}_a\nabla_\delta e^a_\rho \\
& +(6-8c_2-8c_3)T_{\alpha\gamma\beta}e^{\rho}_a\nabla_\delta e^a_\rho+(2+4c_2+12c_3)T_{\alpha\gamma}^{\ \ \mu}e^a_\beta\nabla_\delta e^a_\mu-T_{\alpha\beta}^{\ \ \mu}e^a_\delta\nabla_\gamma e^a_\mu \\
& +(-6-8c_2-16c_3)T_{\delta\rho\beta}e^{\rho}_a\nabla_\alpha e^a_\gamma-(2c_2+4c_3)g_{\alpha\gamma}T_{\mu\beta}^{\ \ \mu}e^{\rho}_a\nabla_\delta e^a_\rho-(2c_2+4c_3)g_{\beta\delta}T_{\alpha\rho}^{\ \ \sigma}e^{\rho}_a\nabla_\gamma e^a_\sigma \\
& -(4+16c_2+32c_3)e^{\mu}_a e_{b\beta}\nabla_\gamma e^a_\alpha\nabla_\delta e^b_\mu+(4+12c_2+32c_3)e_{\delta a}e^{\mu}_b\nabla_\alpha e^a_\gamma\nabla_\beta e^b_\mu \\
& -(2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_a e^{\nu}_b\nabla_\gamma e^a_\mu\nabla_\alpha e^b_\nu+(2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_a e^{\rho}_c\nabla_\alpha e^a_\rho\nabla_\gamma e^c_\mu\Big).
\end{aligned}
\tag{20}
$$

Properties:

-in the zeroth order the action (20) reduces to EH action and the cosmological constant term (arbitrary, constants c_2 and c_3).

-symmetries: NC generalization preserves local Lorentz symmetry but breaks the diffeomorphism symmetry. For example:
-$\nabla_\alpha e_\gamma^a$ can be written (using the metricity condition) as

$$\nabla_\alpha e_\gamma{}^a = \partial_\alpha e_\gamma{}^a + \omega_\alpha^{ab} e_{\gamma b} = \Gamma_{\alpha\gamma}^\sigma e_\sigma{}^a \tag{21}$$

-opservation: models with $SO(1,3)$ gauge symmetry cannot have a correction term of the form $\theta^{\alpha\beta}\theta^{\gamma\delta}g_{\alpha\gamma}g_{\beta\delta}$ (important for NC Minkowski corrections). Consequence of having the gauge field (not the vierbein) as the fundamental field.
-equations of motion: variation with respect to the vierbeins and the spin connection

$$
\begin{aligned}
\delta e_\mu^a : \qquad & R_{\alpha\gamma}{}^{cd} e_d^\gamma e_a^\alpha e_c^\mu - \frac{1}{2} e_a^\mu R + \frac{3}{l^2}(1 + c_2 + 2c_3) e_a^\mu \\
& = \tau_a{}^\mu = -\frac{8\pi G_N}{e} \frac{\delta S_{NC}^{(2)}}{\delta e_\mu^a}, \qquad (22) \\
\delta \omega_\mu^{ab} : \qquad & T_{ac}{}^c e_b^\mu - T_{bc}{}^c e_a^\mu - T_{ab}{}^\mu \\
& = S_{ab}{}^\mu = -\frac{16\pi G_N}{e} \frac{\delta S_{NC}^{(2)}}{\delta \omega_\mu^{ab}}. \qquad (23)
\end{aligned}
$$

NC corrections to Minkowski space-time

Minkowski space-time is a solution of vacuum Einstein equations without the cosmological constant $(1 + c_2 + 2c_3 = 0)$. We are interested in corrections to this solution induced by our NC gravity model.

We assume that the NC metric is of the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu}$ is a small correction that is second order in the deformation parameter $\theta^{\mu\nu}$.

The equation for $h_{\mu\nu}$ is

$$\begin{aligned} & \frac{1}{2}(\partial_\sigma\partial^\nu h^{\sigma\mu} + \partial_\sigma\partial^\mu h^{\sigma\nu} - \partial^\mu\partial^\nu h - \Box h^{\mu\nu}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h) \\ = \; & \frac{11}{4l^6}\left(2\eta_{\alpha\gamma}\theta^{\alpha\mu}\theta^{\gamma\nu} + \frac{1}{2}\eta_{\alpha\gamma}\eta_{\beta\delta}\eta^{\mu\nu}\theta^{\alpha\beta}\theta^{\gamma\delta}\right). \end{aligned} \tag{24}$$

The RHS of equation (24) is constant. Therefore, these equations are solved by a general $h_{\mu\nu}$ quadratic in coordinates. A solution of the form:

$$\begin{aligned} g_{00} &= 1 - \frac{11}{2l^6}\theta^{0m}\theta^{0n}x^m x^n - \frac{11}{8l^6}\theta^{\alpha\beta}\theta_{\alpha\beta}r^2 \\ g^{0i} &= -\frac{11}{3l^6}\theta^{0m}\theta^{0n}x^m x^n, \\ g_{ij} &= -\delta_{ij} - \frac{11}{6l^6}\theta^{im}\theta^{jn}x^m x^n \\ &\quad + \frac{11}{24l^6}\delta^{ij}\theta^{\alpha\beta}\theta_{\alpha\beta}r^2 - \frac{11}{24l^6}\theta^{\alpha\beta}\theta_{\alpha\beta}x^i x^j\,. \end{aligned} \tag{25}$$

Scalar curvature of this solution is $R = -\frac{11}{l^6}\theta^{\alpha\beta}\theta^{\gamma\delta}\eta_{\alpha\gamma}\eta_{\beta\delta} = const.$, (A)dS-like solution. Curvature is induced by the noncommutativity.

The Reimann tensor for this solution can be calculated easily. A very interesting (and unexpected) observation follows: knowing the components of the Riemann tensor the components of the metric tensor can be written as

$$\begin{aligned} g_{00} &= 1 - R_{0m0n}x^m x^n \,, \\ g_{0i} &= -\frac{2}{3} R_{0min}x^m x^n \,, \\ g_{ij} &= -\delta_{ij} - \frac{1}{3} R_{imjn}x^m x^n \,. \end{aligned} \tag{26}$$

This shows that the coordinates x^μ we started with, are Fermi normal coordinates.

Riemann normal coordinates: inertial coordinates in a point, can be constructed in a small neighborhood of that point.
Fermi normal coordinates: inertial coordinates of a local observer moving along some geodesic; can be constructed in a small neighborhood along the geodesic (cylinder), [Manasse, Misner '63; Chicone, Mashoon'06; Klein, Randles '11].

The measurements performed by the local observer moving along the geodesic are described in the Fermi normal coordinates. Especially, he is the one that measures $\theta^{\mu\nu}$ to be constant! In any other reference frame, observers will measure $\theta^{\mu\nu}$ different from constant.

The breaking of diffeomorphism invariance is now understood better: there is a preferred reference system defined by the Fermi normal coordinates and the NC parameter $\theta^{\mu\nu}$ is constant in that particular reference system. The breaking Diff. symmetry can be a consequence of fixing the coordinate system.

Let y^α be an arbitrary coordinate system at a point P in a small neighborhood of the geodesic γ which defines our FNC x^μ and $[x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}$. Then the noncommutativity in y-coordinates is given by

$$[y^\alpha \stackrel{\star}{,} y^\beta] = i\theta^{\mu\nu}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} - \frac{i}{24}\theta^{\mu\nu}\theta^{\rho\sigma}\theta^{\kappa\lambda}\frac{\partial^3 y^\alpha}{\partial x^\kappa \partial x^\rho \partial x^\mu}\frac{\partial^3 y^\beta}{\partial x^\lambda \partial x^\sigma \partial x^\nu} + \dots \; . \tag{27}$$

The $\star$-product is the Moyal-Weyl $\star$-product and y^α are understood as functions of FNC x^μ.

Following closely the notation of [Poisson, Pound, Vega, arXiv:1102.0529][1] we calculate

$$\begin{aligned}\frac{\partial y^\beta}{\partial x^0} &= \bar{e}_0^\beta - \frac{1}{2}\bar{e}_A^\beta R^A{}_{i0j}x^i x^j + \dots \\ \frac{\partial y^\beta}{\partial x^k} &= \bar{e}_k^\beta - \frac{1}{6}\bar{e}_A^\beta R^A{}_{ikj}x^i x^j + \dots . \end{aligned} \tag{28}$$

Here $\bar{e}_A^\beta$ are vierbeins relating coordinates y^α and locally flat coordinates in the given point P and $R^A{}_{ikj}$ and $R^A{}_{i0j}$ are components of the Riemann tensor calculated at the geodesic γ (depending only on the affine parameter t along γ.). Equations (28) contain terms that are higher power in coordinates x^i (indices $i, j, k, \dots$ are spacial indices) and derivatives of Riemann tensor. We only wrote the first approximation.

[1] Note that we use the $(+,-,-,-)$ signature.

Using (28) we calculate the first term in (27):

$$
\begin{aligned}
i\theta^{\mu\nu}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} \quad = \quad & i\Big(\theta^{0i}(\bar{e}_0^\alpha \bar{e}_i^\beta - \bar{e}_i^\alpha \bar{e}_0^\beta) + \theta^{ij}\bar{e}_i^\alpha \bar{e}_j^\beta\Big) \\
& -\frac{i}{6}\theta^{0k}(\bar{e}_0^\alpha \bar{e}_A^\beta - \bar{e}_A^\alpha \bar{e}_0^\beta)R^A{}_{ikj}x^i x^j \\
& -\frac{i}{6}\theta^{kl}(\bar{e}_k^\alpha \bar{e}_A^\beta - \bar{e}_A^\alpha \bar{e}_k^\beta)R^A{}_{ilj}x^i x^j \\
& +\frac{i}{2}\theta^{0k}(\bar{e}_k^\alpha \bar{e}_A^\beta - \bar{e}_A^\alpha \bar{e}_k^\beta)R^A{}_{i0j}x^i x^j + \ldots . \quad (29)
\end{aligned}
$$

Once the explicit form of y^α (in terms of FNC x^μ) is given, one can calculate $[y^\alpha \stackrel{\star}{,} y^\beta]$ more explicitly.

Dirac field coupled to gravity in NC $SO(2,3)_\star$ model

Commutative action:

$$
\begin{aligned}
S &= \frac{i}{12}\int d^4x\, \varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi - D_\sigma\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\psi\Big] \\
&+ \frac{i}{144}\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\bar\psi\Big\{D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi \\
&- D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi + D_\mu\phi D_\nu\phi\phi D_\rho\phi \star D_\sigma\phi\Big\}\psi + h.c.
\end{aligned}
$$

After breaking the symmetry we arrive at

$$
S = \int d^4x\, e\, \frac{i}{2}\left[\bar\psi\gamma^\sigma\nabla_\sigma\psi - \nabla_\sigma\bar\psi\gamma^\sigma\psi\right] - m\int d^4x\, e\, \bar\psi\psi\ , \quad (30)
$$

which is exactly the Dirac action in curved space-time for spinors of mass m.

NC deformation

The noncommutative version of the kinetic action is

$$S_{NC} = \frac{i}{12} \int d^4x \, \varepsilon^{\mu\nu\rho\sigma} \left[\widehat{\bar{\psi}} \star (D_\mu \widehat{\phi}) \star (D_\nu \widehat{\phi}) \star (D_\rho \widehat{\phi}) \star (D_\sigma \widehat{\psi}) \right. \\ \left. -(D_\sigma \widehat{\bar{\psi}}) \star (D_\mu \widehat{\phi}) \star (D_\nu \widehat{\phi}) \star (D_\rho \widehat{\phi}) \star \widehat{\psi} \right] . \tag{31}$$

NC correction of kinetic action in the first order at θ is

$$
\begin{aligned}
S^{(1)}_{kin} = \frac{i}{12}\,\theta^{\alpha\beta}\int d^4x\,\varepsilon^{\mu\nu\rho\sigma}\Big[& - \frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu D_\rho\phi D_\sigma\psi \\
& + \frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi) \\
& + \frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi \\
& + \frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi \\
& + \frac{1}{2}\bar{\psi}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi \\
& + \frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi D_\sigma\psi \\
& + \frac{i}{2}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\psi \\
& - \frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi \Big] + h.c.\,. \qquad (32)
\end{aligned}
$$

This action possesses ordinary $SO(2,3)$, i.e. AdS symmetry. Taking $\phi^a = 0$ and $\phi^5 = l$, this symmetry is broken down to the local Lorentz symmetry. After the symmetry breaking, the first order kinetic term becomes

$$
\begin{aligned}
S^{(1)}_{kin} \quad = \quad & \theta^{\alpha\beta} \int d^4x\, e \Big[-\frac{1}{8} R_{\alpha\mu}{}^{ab} e^{\mu}_{a}\ \bar\psi \gamma_b \nabla_\beta \psi + \frac{1}{16} R_{\alpha\beta}{}^{ab} e^{\sigma}_{b}\ \bar\psi \gamma_a \nabla_\sigma \psi \\
- \quad & \frac{i}{32} R_{\alpha\beta}{}^{ab} \varepsilon_{abc}{}^{d} e^{\sigma}_{d}\ \bar\psi \gamma^c \gamma^5 \nabla_\sigma \psi - \frac{i}{16} R_{\alpha\mu}{}^{bc} e^{\mu}_{a} \varepsilon^{a}{}_{bcm}\ \bar\psi \gamma^m \gamma^5 \nabla_\beta \psi \\
- \quad & \frac{i}{24}\ R_{\alpha\mu}{}^{ab} \varepsilon_{abc}{}^{d} e^{c}_{\beta} (e^{\mu}_{d} e^{\sigma}_{s} - e^{\mu}_{s} e^{\sigma}_{d})\ \bar\psi \gamma^s \gamma^5 \nabla_\sigma \psi \\
- \quad & \frac{i}{8l} T_{\alpha\beta}{}^{a} e^{\sigma}_{a}\ \bar\psi \nabla_\sigma \psi + \frac{i}{8l} T_{\alpha\mu}{}^{a} e^{\mu}_{a}\ \bar\psi \nabla_\beta \psi \\
+ \quad & \frac{1}{16l} T_{\alpha\beta}{}^{a} e^{\mu}_{a}\ \bar\psi \sigma_\mu{}^\sigma \nabla_\sigma \psi + \frac{1}{8l} T_{\alpha\mu}{}^{a} e^{\mu}_{b}\ \bar\psi \sigma_a{}^b \nabla_\beta \psi \\
- \quad & \frac{1}{4} (\nabla_\alpha e^{a}_{\mu})(e^{\mu}_{a} e^{\sigma}_{b} - e^{\sigma}_{a} e^{\mu}_{b})\ \bar\psi \gamma^b \nabla_\beta \nabla_\sigma \psi - \frac{1}{4l} \bar\psi \sigma_\alpha{}^\sigma \nabla_\beta \nabla_\sigma \psi \\
- \quad & \frac{i}{8} \eta_{ab} (\nabla_\alpha e^{a}_{\mu})(\nabla_\beta e^{b}_{\nu}) \varepsilon^{cdrs} e^{\mu}_{c} e^{\nu}_{d} e^{\sigma}_{s}\ \bar\psi \gamma_r \gamma_5 \nabla_\sigma \psi \\
+ \quad & \frac{i}{12} (\nabla_\alpha e^{a}_{\mu})(\nabla_\beta e^{b}_{\nu}) \varepsilon_b{}^{cds} e^{\mu}_{c} e^{\nu}_{d} e^{\sigma}_{s}\ \bar\psi \gamma_a \gamma_5 \nabla_\sigma \psi \\
- \quad & \frac{1}{12l} e^{c}_{\alpha} (\nabla_\beta e^{b}_{\nu}) \varepsilon_{bc}{}^{ds} e^{\nu}_{d} e^{\sigma}_{s}\ \bar\psi \gamma_5 \nabla_\sigma \psi \\
- \quad & \frac{1}{8l} (\nabla_\alpha e^{a}_{\mu})(e^{\mu}_{a} e^{\sigma}_{b} - e^{\sigma}_{a} e^{\mu}_{b}) e^{c}_{\beta}\ \bar\psi \sigma^b{}_c \nabla_\sigma \psi \\
- \quad & \frac{i}{2l} (\nabla_\alpha e^{a}_{\mu}) e^{\mu}_{a}\ \bar\psi \nabla_\beta \psi - \frac{1}{8l} (\nabla_\alpha e^{a}_{\mu}) e^{\mu}_{b}\ \bar\psi \sigma_a{}^b \nabla_\beta \psi \\
+ \quad & \frac{1}{96l} R_{\alpha\beta}{}^{ab}\ \bar\psi \sigma_{ab} \psi - \frac{5}{48l} R_{\alpha\mu}{}^{ab} e^{\mu}_{a} e^{c}_{\beta}\ \bar\psi \sigma_{bc} \psi - \frac{1}{16l} R_{\alpha\mu}{}^{ab} e_{\beta a} e^{\mu}_{c}\ \bar\psi \sigma_b{}^c \psi \\
- \quad & \frac{3}{32l^2} T_{\alpha\beta}{}^{a}\ \bar\psi \gamma_a \psi - \frac{1}{16l^2} T_{\alpha\mu}{}^{a} e^{\mu}_{a}\ \bar\psi \gamma_\beta \psi \\
+ \quad & \frac{1}{16l^2} T_{\alpha\mu}{}^{a} e_{\beta a}\ \bar\psi \gamma^\mu \psi + \frac{1}{12l} \eta_{ab} (\nabla_\alpha e^{a}_{\mu})(\nabla_\beta e^{b}_{\nu})\ \bar\psi \sigma^{\mu\nu} \psi
\end{aligned}
$$

NC corrections to the mass term are calculated similarly.
The NC action is invariant under $SO(1,3)$ transformation and the charge conjugation.
There is a modification of Dirac equation in flat space-time.

$$
\begin{aligned}
S_{NC} = \int d^4x\, \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \theta^{\alpha\beta}\int d^4x \Big[& -\frac{1}{2l}\bar{\psi}\sigma_\alpha{}^\sigma\partial_\beta\partial_\sigma\psi \\
& + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\bar{\psi}\gamma_\rho\gamma_5\partial_\sigma\psi - M\bar{\psi}\sigma_{\alpha\beta}\psi\Big] . \qquad (34)
\end{aligned}
$$

$$\left[i\partial\!\!\!/ - m - \frac{1}{2l}\theta^{\alpha\beta}\sigma_{\alpha}{}^{\sigma}\partial_{\beta}\partial_{\sigma} + \frac{7i}{24l^2}\theta^{\alpha\beta}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_{\rho}\gamma_5\partial_{\sigma} - \theta^{\alpha\beta}M\sigma_{\alpha\beta}\right]\psi = 0\,. \tag{35}$$

To simplify further analysis, we will assume $\theta^{12} = -\theta^{21} =: \theta \neq 0$ and all other components of $\theta^{\mu\nu}$ equal to zero.

We consider an electron moving along $z-$ direction.

Dispersion relation:

$$\begin{aligned} E_{1,2} &= E_{\mathbf{p}} \mp \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} + \mathcal{O}(\theta^2)\ , \\ E_{3,4} &= -E_{\mathbf{p}} \pm \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} + \mathcal{O}(\theta^2)\ , \end{aligned} \tag{36}$$

$$\psi_1 \sim \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1+\left(\frac{m}{12l^2}-\frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \end{pmatrix} e^{-iE_1 t+ip_z z} \ ,$$

$$\psi_2 \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1-\left(\frac{m}{12l^2}-\frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \end{pmatrix} e^{-iE_2 t-ip_z z} \ . \qquad (37)$$

The solutions with oposite helicity have different energies. For positive (negative) helicity solution we get

$$\mathbf{v}_{1,2} = \frac{\mathbf{p}}{E_{\mathbf{p}}}\left[1 \pm \left(\frac{m^2}{12l^2} - \frac{m}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}^2} + \mathcal{O}(\theta^2)\right] \ . \qquad (38)$$

These velocities can be rewritten in the following way:

$$\mathbf{v}_{1,2} = \frac{\mathbf{p}}{E_{1,2}} + \mathcal{O}(\theta^2) \ . \qquad (39)$$

Velocity of an electron moving in z-direction depends on its helicity. This is analogues to the birefringence effect.

Electromagnetic field

Gauge group: $SO(2,3) \times U(1)$.
Gauge potential

$$\Omega_\mu = \omega_\mu + A_\mu \ . \tag{40}$$

The field strength associated with the gauge potential Ω_μ is

$$\mathbb{F}_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu - i[\Omega_\mu, \Omega_\nu] \ , \tag{41}$$

and it can be decomposed as

$$\mathbb{F}_{\mu\nu} = F_{\mu\nu} + \mathcal{F}_{\mu\nu} \ , \tag{42}$$

The action:

$$\begin{aligned} S_A &= -\frac{1}{16l}\mathrm{Tr}\int \mathrm{d}^4x \ \epsilon^{\mu\nu\rho\sigma}\Big(f\mathbb{F}_{\mu\nu}D_\rho\phi D_\sigma\phi\phi \\ &+ \frac{i}{3!}ffD_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\Big) + h.c. \, . \end{aligned} \tag{43}$$

The action (43) includes an additional auxiliary field f

$$f = \frac{1}{2}f^{AB}M_{AB} \ , \quad \delta_\varepsilon f = i[\varepsilon, f] \tag{44}$$

After SB and elimination of auxiliary field we get

$$S_A = -\frac{1}{4}\int \mathrm{d}^4x \; e \; g^{\mu\rho} g^{\nu\sigma} \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} \,. \tag{45}$$

Full NC action

$$\widehat{S} = \widehat{S}_A + \widehat{S}_\psi \ , \tag{46}$$

where

$$\begin{aligned} \widehat{S}_A \;&=\; -\frac{1}{16l}\mathrm{Tr}\int \mathrm{d}^4x \; \epsilon^{\mu\nu\rho\sigma}\Big(\hat{f}\star\hat{\mathbb{F}}_{\mu\nu}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\phi} \\ &\quad +\frac{i}{3!}\hat{f}\star\hat{f}\star D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\phi}\Big) + h.c. \,. \end{aligned} \tag{47}$$

$$\widehat{S}_\psi = \ldots \tag{48}$$

The action for NC electrodynamics in flat space-time up to the first order in $\theta^{\alpha\beta}$ is given by

$$\begin{aligned}\widehat{S}_{flat} = &\int \mathrm{d}^4x\, \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}\int d^4x\, \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \\ &+ \theta^{\alpha\beta}\int \mathrm{d}^4x\, \Big(\frac{1}{2}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}\mathcal{F}^{\mu\nu} - \frac{1}{8}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\Big) \\ &+ \theta^{\alpha\beta}\int \mathrm{d}^4x\, \bar{\psi}\Bigg(- \frac{1}{2l}\sigma_\alpha{}^\sigma \mathcal{D}_\beta\mathcal{D}_\sigma + \frac{7i}{24l^2}\epsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\mathcal{D}_\sigma \\ &- \left(\frac{m}{4l^2} + \frac{1}{6l^3}\right)\sigma_{\alpha\beta} + \frac{3i}{4}\mathcal{F}_{\alpha\beta}\not{D} \\ &- \frac{i}{2}\mathcal{F}_{\alpha\mu}\gamma^\mu\mathcal{D}_\beta - \left(\frac{3m}{4} - \frac{1}{4l}\right)\mathcal{F}_{\alpha\beta}\Bigg)\psi\,, \end{aligned} \tag{49}$$

The equation of motion:

$$\left(i\partial\!\!\!/ - m + A\!\!\!/ + \theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}\right)\psi = 0\,, \tag{50}$$

where $\theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}$ is given by

$$\begin{aligned}\mathcal{M}_{\alpha\beta} = &-\frac{1}{2l}\sigma_{\alpha}{}^{\sigma}\mathcal{D}_{\beta}\mathcal{D}_{\sigma} + \frac{7i}{24l^2}\epsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_{\rho}\gamma_5\mathcal{D}_{\sigma} - \left(\frac{m}{4l^2} + \frac{1}{6l^3}\right)\sigma_{\alpha\beta} \\ &-\frac{i}{2}\mathcal{F}_{\alpha\mu}\gamma^{\mu}\mathcal{D}_{\beta} + \frac{1}{4l}\mathcal{F}_{\alpha\beta}\,.\end{aligned} \tag{51}$$

NC Landau problem

Phenomenological consequences of our model and NC in general: the NC Landau problem: an electron moving in the x-y plane in the constant magnetic field $\vec{B} = B\vec{e}_z$.
For simplicity: $\theta^{12} = \theta \neq 0$ and $A_\mu = (0, By, 0, 0)$. Assume

$$\psi = \begin{pmatrix} \varphi(y) \\ \chi(y) \end{pmatrix} e^{-iEt + ip_x x + ip_z z} . \tag{52}$$

with φ, χ and E represented as powers series in θ.

Deformed energy levels, i.e., NC Landau levels are given by

$$E_{n,s} = E^{(0)}_{n,s} + E^{(1)}_{n,s}\,, \tag{53}$$
$$E^{(0)}_{n,s} = \sqrt{p_z^2 + m^2 + (2n+s+1)B}\,,$$
$$E^{(1)}_{n,s} = -\frac{\theta s}{E^{(0)}_{n,s}}\left(\frac{m^2}{12l^2} - \frac{m}{3l^3}\right)\left(1 + \frac{B}{(E^{(0)}_{n,s}+m)}(2n+s+1)\right) + \frac{\theta B^2}{2E^{(0)}_{n,s}}(2n+s+1)\,.$$

Here $s = \pm 1$ is the projection of electron spin. In the nonrelativistic limit and with $p_z = 0$, (53) reduces to

$$E_{n,s} = m - s\theta\left(\frac{m}{12l^2} - \frac{1}{3l^3}\right) + \frac{2n+s+1}{2m}B_{eff} - \frac{(2n+s+1)^2}{8m^3}B^2_{eff} + \mathcal{O}(\theta^2)\,, \tag{54}$$
$$B_{eff} = (B + \theta B^2)\,.$$

Consistent with string theory interpretation of noncommutativity as a Neveu-Schwarz B-field.

In addition, the induced magnetic dipole moment of an electon is given by

$$\mu_{n,s} = -\frac{\partial E_{n,s}}{\partial B} = -\mu_B(2n + s + 1)(1 + \theta B)\,, \qquad (55)$$

where $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton.

Some numbers:

-$\theta = \frac{\hbar^2 c^2}{\Lambda_{NC}^2}$ and $\Lambda_{NC} \sim 10\,TeV$,

-accuracy of magnetic moment measurements $\delta\mu_{n,s} \sim 10^{-13}$,

-for observable effects in $\mu_{n,s}$, $B \sim 10^{11}\,T$ needed. This is the magnetic field of some neutron stars (magnetars), in laboratory $B \sim 10^3\,T$.

Conclusion

NC gravity: a general NC $SO(2,3)_\star$ action studied, expansion up to second order in the NC parameter written in a manifestly gauge covariant way;

NC corrections to Minkowski space-time

-solution of the NC Einstein equations

-emergence of Fermi normal coordinates

-better understanding of θ-constant noncomutativity

-Coupling Dirac and electromagnetic field with gravity