The action for scalar, Dirac and gauge field as $3-BF$ action with constraints

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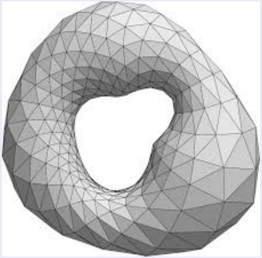
Outline

The fundamental principle

- Spacetime is a piecewise linear manifold - it's not just a **regulatar**, but a **physical entity**.
- Field theory reconstructed only as an approximation, like in fluid mechanics.
- Nature has a **physical cutoff** - there is a notion of the smallest possible lenght.
- The configuration integral is defined by discretization:

$$Z = \sum_{\{\phi\}} \prod_{v\in T} \mathcal{A}_v(\phi) \prod_{\epsilon\in T} \mathcal{A}_\epsilon(\phi) \cdots \prod_{\sigma\in T} \mathcal{A}_\sigma(\phi),$$



$\mathcal{T}(\mathcal{M}_D)$ triangulation of a manifold $\mathcal{M}_D$

- Finite number of degrees of freedom (in a finite volume): D-simplices are flat, curvature is obtained by non-trivial joining of simplices - Regge calculus
- Every object is colored with ϕ - fundamental variable, and amplitude $\mathcal{A}$ - describing dynamics of ϕ.

Quantization procedure

- Quantization using state sums:
  - Rewrite the gravitational action as a sum of the topological part of the action and the residual part:

Gravity

- topological part

$$S_{BFCG} = \int_M (B_{ab} \wedge R^{ab} + e_a \wedge \nabla \beta^a)$$

- topological part + constraint

$$S_{GR} = \int B_{ab} \wedge R^{ab} + e_a \wedge G^a - \phi^{ab} \wedge (B_{ab} - \frac{1}{16\pi l_p^2} \epsilon_{abcd} e^c \wedge e^d)$$

  - Construct a state sum for the topological sector of the theory, using the topological quantum field theory formalism:

$$Z^{disc} = \int \left(\prod_\epsilon \mathrm{d}L_\epsilon \right) \sum_{\{\Lambda_\Delta\}} \sum_{\{I_\tau\}} W(L, \Lambda, I)$$

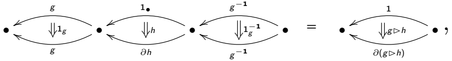
Quantization procedure

- This sum is a topological invariant independent of the triangulation.
- The integral measure is defined.
- Modification of the amplitudes in a certain way provides the transition from the topological state sum to the state sum corresponding to the complete theory.

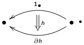
Gravity

Category theory

- A *category* consists of the elements called *objects* and *morphisms* - the mappings between these objects - a *group* is then regarded as a category with only one object, where all morphisms are invertible.
- The 2-generalization of the notion of category - a 2-*category* consists of: a collection of *objects*, *morphisms*, and 2-*morphisms* (2-*group*),
- 2-group is equivalent to a **crossed-module** $(G, H, \triangleright, \partial)$:
    - Lie group G - elements of the group being morphisms and the group operation being composition of these morphisms,
    - Lie group H - contains 2-morphisms whose source is the identity, where horizontal composition is the group operation,
    - horizontal conjugation of each element $h \in H$ by the element $g \in G$, i.e. an action of group G on H $\triangleright : G \to Aut(H)$



    - a group homomorphism that maps every 2-morphism in H to a target in G, $\partial : H \to G$:



Poincaré 2-group

- A crossed-module $(G, H, \triangleright, \partial)$, where:
  - $G = SO(1,3)$,
  - $H = \mathbb{R}^4$,
  - $\triangleright$ is a representation of the group G on H: $SO(1,3) \times \mathbb{R}^4 \to \mathbb{R}^4$,
  - ∂ is trivial, i.e. every object $h \in H$ is mapped to the identity element in G: $\mathbb{R}^4 \to 1_{SO(1,3)}$.
- Besides the connection $\mathfrak{so}(1,3)$-albegra-valued 1-form $\omega \in \mathfrak{g}$, there is a 2-connection given by the pair (ω, β), where $\beta \in \mathfrak{h}$ is an $\mathbb{R}^4$-albegra-valued 2-form.
- Connections transforms under G-gauge transformations:

$$\omega \to g^{-1}\omega g + g^{-1}\mathrm{d}g, \quad \beta \to g^{-1} \triangleright \beta\,,$$

  and under H-gauge transformations:

$$\omega \to \omega + \partial\eta \quad \text{i} \quad \beta \to \beta + d\eta + \omega \wedge^{\triangleright} \eta + \eta \wedge \eta.$$

- We define holonomy g_l and 2-holonomy h_f:

$$g_l = \exp\left(\int_l \omega\right) \in G, \quad h_f = \exp\left(\int_f \beta\right) \in H.$$

Gravitation

$2-BF$ action with constraints

- Gravity is obtained as $2-BF$ action with constraints:

$$S_{GR} = \int B_{ab} \wedge R^{ab} + e_a \wedge G^a - \phi^{ab} \wedge (B_{ab} - \frac{1}{16\pi l_p^2} \epsilon_{abcd} e^c \wedge e^d)$$

- Equations of motion ($B, e, \omega, \beta, \phi$):

$$R_{ab} - \phi_{ab} = 0 \tag{1}$$

$$\nabla \beta_a + \frac{1}{8\pi l_p^2} \epsilon_{abcd} \phi^{bc} \wedge e^d = 0 \tag{2}$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} = 0 \tag{3}$$

$$\nabla e_a = 0 \tag{4}$$

$$B_{ab} - \frac{1}{16\pi l_p^2} \epsilon_{abcd} e^c \wedge e^d \tag{5}$$

$$\Rightarrow \beta = 0$$

$$\Rightarrow \epsilon^{abcd} R_{bc} \wedge e^d = 0$$

Gauge field

2-group

- Crossed-module $(G, H, \triangleright, \partial)$, where:
  - $G = SO(1,3)xSU(N)$,
  - $H = \mathbb{R}^4$,
  - $\triangleright$ is a representation of the group G on H,
  - ∂ is trivial: $\mathbb{R}^4 \to 1_{SO(1,3)xSU(N)}$.
- Covariant derivative and curvature

$$\mathcal{F} = \mathrm{d}\omega + \omega \wedge \omega + \mathrm{d}A + A \wedge A$$

- Bilinear form $\langle , \rangle_{\mathfrak{g}}$ such that $B \wedge F = B_{ab} \wedge R^{ab} + B_I \wedge F^I$

Gauge fields

2-BF action and equations of motion

$$S_M = \int B_I \wedge F^I + \lambda^I \wedge (B_I - \alpha M_{abI} e^a \wedge e^b) + \zeta^{abI}(M_{abI}\epsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l - g_{IJ} F^J \wedge e_a \wedge e_b),$$

$$M_{abI} = -\frac{1}{\alpha}\zeta_{abI}$$

$$B_{I\alpha\beta} = 2\alpha M_{abI} e^a_\alpha e^b_\beta$$

$$\zeta^{abI} = \frac{e}{48}\alpha \epsilon^{\alpha\beta\gamma\delta} F^I_{\alpha\beta} e^a_\gamma e^b_\delta$$

Equations of motion:

- $-\mathrm{d}B_I + B_A \wedge C^A{}_{JI} A^J + \mathrm{d}(\zeta^{ab}_I e_a \wedge e_b) - \zeta^{ab}_A e_a \wedge e_b \wedge C^A{}_{JI} A^J = 0$
- $\nabla \beta_a + \frac{1}{8\pi l_p^2}\epsilon_{abcd}\phi^{bc} \wedge e^d - 2\alpha M_{abI}\lambda^I \wedge e^b + 4\zeta^{rsI} M_{rsI}\epsilon_{ajkl} e^j \wedge e^k \wedge e^l - 2\zeta_{ab}{}^I F_I \wedge e^b = 0$

$$\boxed{R^{\sigma\rho} - \frac{1}{2} g^{\sigma\rho} R = 8\pi l_p^2 \left(-\frac{1}{4}(F^I_{\mu\nu} F_I^{\mu\nu} g^{\rho\sigma} + 4 F_I^{\sigma\nu} F^I{}_\nu{}^\rho)\right)}, \ \alpha = \frac{12}{g},$$

$$\boxed{(\nabla_\rho F^{\rho\alpha})_I = 0}.$$

Scalar field

3-group

- 3-category consists of *objects*, *morphisms*, 2-*morphisms* and 3-*morphisms*.
- 3-group is equivalent to 2-**crossed-module**
  - Lie groups L, G and H,
  - group homomorphisms ∂ and δ:

$$L \xrightarrow{\delta} E \xrightarrow{\partial} G \,,$$

  - an action $\triangleright$ of the group G on groups L and E,
  - G-eqiuvariant function $\{,\} : E \times E \to L$, that has certain properties.
- Lie algebra-valued differential forms $\omega \in \mathcal{A}^1(M,\mathfrak{g})$, $\beta \in \mathcal{A}^2(M,\mathfrak{e})$ and $\gamma \in \mathcal{A}^3(M,\mathfrak{l})$,
- 2-curvature 3-form (ω,β): $\Omega = \mathrm{d}\omega + \omega \wedge \omega$, curvature of ω, and $\Gamma = \mathrm{d}\beta + \omega \wedge^{\triangleright} \beta$, covariant derivative of β,
- 3-curvature 4-form (ω,β,γ): $\Theta = d\gamma + \omega \wedge^{\triangleright} \gamma - \beta \wedge^{\{,\}} \beta$.

Scalar field

Action and equations of motion

- Action for scalar field

$$S_{scalar} = \int \phi \, \mathrm{d}B + \chi \wedge (B - \gamma H_{abc} e^a \wedge e^b \wedge e^c) + \Lambda^{ab} \wedge (H_{abc} \epsilon^{cdef} e_d \wedge e_e \wedge e_f - F \wedge e_a \wedge e_b) \\ - \frac{1}{4!} \gamma m^2 \phi^2 \epsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l$$

- Equations of motion:

- $\mathrm{H}_{abc} = \frac{1}{3! e} \epsilon_{\alpha\beta\gamma\delta} F^\alpha e_a^\beta e_b^\gamma e_c^\delta$
- $\chi = F$
- $\Lambda_{ab\alpha} = \frac{1}{3! e} \gamma \epsilon_{\alpha\beta\gamma\delta} F^\beta e_a^\gamma e_b^\delta$

Equations for motion for ϕ and e^a

- $\mathrm{dB}\text{-}\mathrm{d}(\Lambda^{ab} \wedge e_a \wedge e_b) - \frac{2}{4!} \gamma m^2 \phi \epsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l = 0$
- $$\nabla \beta_a + \frac{1}{8\pi l_p^2} \epsilon_{abcd} \phi^{bc} \wedge e^d + 3\gamma H_{abc} \lambda \wedge e^b \wedge e^c + 3 H^{ijk} \epsilon_{akbc} \Lambda_{ij} \wedge e^b \wedge e^c \\ - 2\Lambda_{ab} \wedge F \wedge e^b - 4 \frac{1}{4!} \gamma m^2 \phi \epsilon_{ajkl} e^j \wedge e^k \wedge e^l =$$

$$\boxed{\partial_\alpha (e F^\alpha) - e m^2 \phi = 0}$$

$$\boxed{R^\delta{}_\gamma - \frac{1}{2} \delta^\delta_\gamma R = F_\gamma F^\delta - \delta^\delta_\gamma \frac{1}{2} \left(F_\alpha F^\alpha - m^2 \phi^2 \right)}$$

Fermionic field

Fermions

Action and equations of motion

- Action

$$S_{fermion} = \int \Big(\psi^{\tilde\alpha}(\gamma_1 \overleftarrow{\nabla})_{\tilde\alpha} + \lambda_1^{\tilde\alpha} \wedge (\gamma_{1\tilde\alpha} - i\kappa_1 \epsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar\psi \gamma^d)_{\tilde\alpha})$$
$$- \bar\psi_{\tilde\alpha}(\overrightarrow{\nabla}\gamma_2)^{\tilde\alpha} - \lambda_{2\tilde\alpha} \wedge (\gamma_2{}^{\tilde\alpha} - i\kappa_1 \epsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^{\tilde\alpha})$$
$$+ i\kappa_1 \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge \bar\psi \frac{im}{2} e^d \psi \Big) + i16\pi l_p^2 \kappa_2 \int \epsilon_{abcd} e^a \wedge e^b \wedge \beta^c \bar\psi \gamma_5 \gamma^d \psi.$$

- Equations of motion:

$$\boxed{R^{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} R = 8\pi l_p^2 \left(\frac{i}{2} \bar\psi (\gamma^\sigma \overleftrightarrow{\nabla}^\rho - \gamma^a \overleftrightarrow{\nabla}_a g^{\sigma\rho} + 2img^{\sigma\rho}) \psi \right)}$$

$$\boxed{(i\gamma^\mu \nabla_\mu - m)\,\psi = 0}$$

Fermions

- Equations of motion for ω_{ab}, β_a, ϕ_{ab}:

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - 2\kappa_2 \epsilon_{abcd} e^c \wedge s^d = 0$$

$$\nabla e_a + 16\pi l_p^2 \kappa_2 s_a = 0$$

$$B_{ab} - \frac{1}{16\pi l_p^2} \epsilon_{abcd} e^c \wedge e^d = 0$$

As in the case without matter $\nabla B^{ab} = -\frac{1}{8\pi l_p^2} \epsilon^{abcd} (e_c \wedge \nabla e_d)$

$$2\, \epsilon_{abcd} e^c \wedge \left(\frac{1}{16\pi l_p^2} \nabla e^d + \kappa_2 s^d \right) + e_{[a} \wedge \beta_{b]} = 0\,.$$

This equaton gives $e_{[a} \wedge \beta_{b]} = 0$, such that $\beta^a = 0$.

- Total action $S_{GR} + S_G + S_F$ gives equations of motion:

$$\boxed{(\nabla_\alpha F^{\alpha\beta})_I = igj_I^\beta}$$

$$\boxed{R^{\sigma\rho} - \frac{1}{2} g^{\sigma\rho} R = 8\pi l_p^2 \Big(-\frac{1}{4g} (F_{\mu\nu}^I F_I^{\mu\nu} g^{\rho\sigma} + 4F_I^{\sigma\nu} F^I{}_\nu{}^\rho) + \frac{i}{2} \bar{\psi} (\gamma^\sigma \overset{\leftrightarrow}{\nabla}{}^\rho - \gamma^a \overset{\leftrightarrow}{\nabla}_a g^{\sigma\rho} - 2img^{\sigma\rho}) \psi \Big)}$$

Total action

$$S = \int \Big[B_{\hat{a}} \wedge \mathcal{F}^{\hat{a}} + e_{\tilde{\alpha}} \wedge \mathcal{G}^{\tilde{\alpha}} + D_{\hat{A}} \wedge \mathcal{H}^{\hat{A}}$$
$$+ \lambda^{\hat{a}} \wedge (B_{\hat{a}} - C_{\hat{a}}{}^{\hat{b}} M_{\hat{b}cd} e^c \wedge e^d) + \lambda^{\hat{A}} \wedge (\gamma_{\hat{A}} - e^a \wedge e^b \wedge e^c C_{\hat{A}}{}^{\hat{B}} M_{abc\,\hat{B}})$$
$$+ \zeta_{\hat{a}}{}^{cd} (M^{\hat{a}}{}_{cd} \epsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l - F^{\hat{a}} \wedge e_c \wedge e_d) + \zeta^{ab}{}_{\hat{A}} \wedge (M_{abc}{}^{\hat{A}} \epsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat{A}} \wedge e_a \wedge e_b)$$
$$- \epsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l (D_{\hat{A}} C^{\hat{A}}{}_{\hat{B}} M^{\hat{B}}{}_{\hat{C}} D^{\hat{C}} + i\kappa_2 \epsilon_{abcd} e^a \wedge e^b \wedge \beta^c \bar{\psi} \gamma_5 \gamma^d \psi \Big],$$

where:

$$B_{\hat{a}} = \begin{bmatrix} B_{ab} & B_I \end{bmatrix}, \quad \mathcal{F}_{\hat{a}} = \begin{bmatrix} R_{ab} & F_I \end{bmatrix}, \quad D_{\hat{A}} = \begin{bmatrix} \phi & \psi_{\tilde{\alpha}} & \bar{\psi}^{\tilde{\alpha}} \end{bmatrix}, \quad \mathcal{H}_{\hat{A}} = \begin{bmatrix} \mathrm{d}\gamma & (\gamma_1 \overleftarrow{\nabla})_{\tilde{\alpha}} & -(\overrightarrow{\nabla}\gamma_2)^{\tilde{\alpha}} \end{bmatrix},$$
$$\gamma_{\hat{A}} = \begin{bmatrix} \gamma & \gamma_{1\tilde{\alpha}} & \gamma_2{}^{\tilde{\alpha}} \end{bmatrix}, \quad \lambda_{\hat{a}} = \begin{bmatrix} -\lambda_{ab} & \lambda_I \end{bmatrix}, \quad M_{\hat{a}cd} = \begin{bmatrix} \epsilon_{abcd} & M_{cd\,I} \end{bmatrix}, \quad \zeta_{\hat{a}}{}^{cd} = \begin{bmatrix} 0 & \zeta^{cd}{}_I \end{bmatrix},$$
$$\lambda_{\hat{A}} = \begin{bmatrix} \lambda & \lambda_{1\tilde{\alpha}} & -\lambda_2{}^{\tilde{\alpha}} \end{bmatrix}, \quad M_{abc\,\hat{A}} = \begin{bmatrix} M_{abc} & \epsilon_{abcd} (\bar{\psi}\gamma^d)_{\tilde{\alpha}} & \epsilon_{abcd} (\gamma^d \psi)^{\tilde{\alpha}} \end{bmatrix}, \quad \zeta_{ab\,\hat{A}} = \begin{bmatrix} \zeta_{ab} & 0 & 0 \end{bmatrix},$$

$$C^{\hat{a}}{}_{\hat{b}} = \begin{bmatrix} G & \\ & \alpha \end{bmatrix}, \quad C^{\hat{A}}{}_{\hat{B}} = \begin{bmatrix} \gamma & & \\ & \kappa_1 & \\ & & \kappa_1 \end{bmatrix}, \quad M^{\hat{A}}{}_{\hat{B}} = \begin{bmatrix} \frac{1}{4!} m_{scal} & & \\ & & m_{fer} \\ & m_{fer} & \end{bmatrix}.$$

Conclusions

- Overview of results:
  - The action for gravitation, gauge, scalar and Fermion fields can be written as $3-BF$ action with constraints.
  - There is a part of the structure of 3-group that corresponds to scalar and fermionic fields, i.e. there is a Lie group associated to matter sector.
- Topics for futher research:
  - Find a 3-group that produce needed matter section.
  - Quantization.

Thank you!

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