University of Belgrade
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Noncommutativity and Nonassociativity of Closed Bosonic String on T-dual Toroidal Backgrounds

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Table of Contents

Introduction

1. Noncommutativity of coordiantes
2. T-duality
3. Generalized Buscher procedure

The action of the closed bosonic string in the presence of background fields is given by

$$S = k\int_\Sigma d^2\xi\sqrt{-g}\Big\{\Big[\frac{1}{2}g^{\alpha\beta}G_{\mu\nu} + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}\Big]\partial_\alpha x^\mu \partial_\beta x^\nu + \frac{\Phi}{4k\pi}R^{(2)}\Big\}. \quad (1)$$

Keeping conformal symmetry at quantum level demands that background fields obey following equations.

$$R_{\mu\nu} - \frac{1}{4}B_{\mu\rho\sigma}B_\nu{}^{\rho\sigma} + 2D_\mu\partial_\nu\Phi = 0, \quad (2)$$

$$D_\rho B^\rho{}_{\mu\nu} - 2\partial_\rho\Phi B^\rho{}_{\mu\nu} = 0, \quad (3)$$

$$4(\partial\Phi)^2 - 4D_\mu\partial^\mu\Phi + \frac{1}{12}B_{\mu\nu\rho}B^{\mu\nu\rho} - R = 0. \quad (4)$$

$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is field strength tensor of Kalb Ramond field.

We will work in $D = 3$ dimensions with following background fields:

$$B_{\mu\nu} = \begin{bmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Final form of our action, in world-sheet light-cone coordinates $\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma)$, $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$. is then:

$$S = k \int_{\Sigma} d^2\xi^{\pm} \Big[Hz(\partial_+ x \partial_- y - \partial_+ y \partial_- x) + \frac{1}{2}(\partial_+ x \partial_- x + \partial_+ y \partial_- y + \partial_+ z \partial_- z) \Big]. \tag{6}$$

T-dualization

T-dualization along x direction

Replacing partial with covariant derivatives

$$\partial_\pm x \to D_\pm x = \partial_\pm x + v_\pm. \tag{7}$$

Removal of unphysical degrees of freedom

$$S_{add} = \frac{k}{2}\int_\Sigma d^2\xi y_1(\partial_+ v_- - \partial_- v_+). \tag{8}$$

Gauge fixing, $x = const$.

$$S_{fix} = k\int_\Sigma d^2\xi[\frac{1}{2}(v_+ v_- + \partial_+ y\partial_- y + \partial_+ z\partial_- z) + v_+ Hz\partial_- y - \partial_+ yHzv_- + \frac{1}{2}y_1(\partial_+ v_- - \partial_- v_+)]. \tag{9}$$

Equations of motion are:

$$F_{+-} = \partial_+ v_- - \partial_- v_+ = 0 \quad \to \quad v_\pm = \partial_\pm x \tag{10}$$

$$v_\pm = \pm\partial_\pm y_1 \pm 2Hz\partial_\pm y \tag{11}$$

Obtaining T-dual action

$$
{}_xS = k\int_\Sigma d^2\xi\Big[\frac{1}{2}(\partial_+ y^1\partial_- y^1 + \partial_+ y\partial_- y + \partial_+ z\partial_- z)
+ \partial_+ y^1 Hz\partial_- y + \partial_+ y Hz\partial_- y^1\Big]. \tag{12}
$$

Getting T-dual transformation laws

$$
\partial_\pm x \cong \pm\partial_\pm y_1 \pm 2Hz\partial_\pm y \quad\rightarrow\quad \dot{x} \cong y_1' + 2Hzy', \tag{13}
$$

$$
\pi_x = \frac{\delta S}{\delta\dot{x}} = k(\dot{x} - 2Hzy') \quad\rightarrow\quad \pi_x \cong ky_1'. \tag{14}
$$

Repeating previous procedure for y coordinate we obtain

$$S_{fix} = k \int_\Sigma d^2\xi \Big[\frac{1}{2}(\partial_+ y_1 \partial_- y_1 + v_+ v_- + \partial_+ z \partial_- z) + \partial_+ y_1 H z v_- + v_+ H z \partial_- y_1 + \frac{1}{2} y_2 (\partial_+ v_- - \partial_- v_+)\Big]. \tag{15}$$

Equation of motion are:

$$\partial_+ v_- - \partial_- v_+ = 0 \rightarrow v_\pm = \partial_\pm y, \tag{16}$$

$$v_\pm = \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1. \tag{17}$$

Twice dualized action is given as

$$ {}_{xy}S = k \int_\Sigma d^2\xi \Big[\frac{1}{2}(\partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + \partial_+ z \partial_- z) + \partial_+ y_2 Hz \partial_- y_1 - \partial_+ y_1 Hz \partial_- y_2 \Big]. \tag{18} $$

T-dual transformation law:

$$ \partial_\pm y \cong \pm \partial_\pm y_2 - 2Hz \partial_\pm y_1 \quad \to \quad \dot{y} \cong y_2' - 2Hz\dot{y}_1, \tag{19} $$

$$ \pi_y = \frac{\delta S}{\delta \dot{y}} = k(\dot{y} + 2Hzx') \quad \to \quad \pi_y \cong k y_2'. \tag{20} $$

Introduction of covariant derivative and invariant coordinate

$$\partial_\pm z \to D_\pm z = \partial_\pm z + v_\pm; \quad z^{inv} = \int_P d\xi^\alpha D_\alpha z = z(\xi) - z(\xi_0) + \Delta V, \tag{21}$$

where

$$\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-). \tag{22}$$

After removing additional degrees of freedom and fixing gauge with $z(\xi) = z(\xi_0)$ we have following action:

$$\begin{aligned} S_{fix} = k \int_\Sigma d^2\xi \Big[& - H\Delta V(\partial_+ y_1 \partial_- y_2 - \partial_+ y_2 \partial_- y_1) \\ & + \frac{1}{2}(\partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + \partial_+ z \partial_- z) \\ & + \frac{1}{2} y_3 (\partial_+ v_- - \partial_- v_+) \Big]. \end{aligned} \tag{23}$$

T-dualization along z direction

Equations of motion are:

$$\partial_+ v_- - \partial_- v_+ = 0 \quad \rightarrow \quad v_\pm = \partial_\pm z, \tag{24}$$

$$v_\pm = \pm\partial_\pm y_3 - 2\beta^\mp, \tag{25}$$

where

$$\beta^\pm = \pm\frac{1}{2}H(y_1\partial_\mp y_2 - y_2\partial_\mp y_1). \tag{26}$$

Fully T-dualized action is:

$$ {}_{xyz}S = k\int_\Sigma d^2\xi\Big[\frac{1}{2}(\partial_+ y_1\partial_- y_1 + \partial_+ y_2\partial_- y_2 + \partial_+ y_3\partial_- y_3) - \partial_+ y_1 H\Delta\bar{y}_3\partial_- y_2 + \partial_+ y_2 H\Delta\bar{y}_3\partial_- y_1\Big]. \tag{27}$$

T-dual transformation laws are:

$$\partial_\pm z \cong \pm\partial_\pm y_3 - 2\beta^\mp \quad \rightarrow \quad \dot{z} \cong y_3' + H(y_1 y_2' - y_2 y_1'), \tag{28}$$

$$\pi_z = \frac{\delta S}{\delta\dot{z}} = k\dot{z} \quad \rightarrow \quad \pi_z \cong k y_3' + kH(xy' - yx'). \tag{29}$$

Noncommutativity and nonassociativity relations

Noncommutativity relations

Transformation laws in canonical form are:

$$y_1{}' = \frac{1}{k}\pi_x, \quad y_2' = \frac{1}{k}\pi_y, \quad y_3' = \frac{1}{k}\pi_z - H(xy' - yx'). \tag{30}$$

Only non-trivial Poisson brackets are:

$$\{y_1(\sigma), y_3(\bar{\sigma})\} \cong -\frac{H}{k}\Big[2y(\sigma) - y(\bar{\sigma})\Big]\theta(\sigma - \bar{\sigma}), \tag{31}$$

$$\{y_2(\sigma), y_3(\bar{\sigma})\} \cong \frac{H}{k}\Big[2x(\sigma) - x(\bar{\sigma})\Big]\theta(\sigma - \bar{\sigma}). \tag{32}$$

For $\sigma - \bar{\sigma} = 2\pi$ we have

$$\{y_1(\sigma + 2\pi), y_3(\sigma)\} \cong -\frac{H}{k}\Big[4\pi N_y + y(\sigma)\Big], \tag{33}$$

$$\{y_2(\sigma + 2\pi), y_3(\sigma)\} \cong \frac{H}{k}\Big[4\pi N_x + y(\sigma)\Big], \tag{34}$$

where N_x and N_y are winding numbers defined as

$$x(\sigma + 2\pi) - x(\sigma) = 2\pi N_x, \quad y(\sigma + 2\pi) - y(\sigma) = 2\pi N_y \tag{35}$$

Nonassociativity relation is defined as

$$\begin{aligned}
&\{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \\
&\{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} + \{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\} \\
&\cong -\frac{2H}{k^2}\Big[\theta(\sigma_1 - \sigma_2)\theta(\sigma_2 - \sigma_3) \\
&+ \theta(\sigma_2 - \sigma_1)\theta(\sigma_1 - \sigma_3) + \theta(\sigma_1 - \sigma_3)\theta(\sigma_3 - \sigma_2)\Big]. \qquad (36)
\end{aligned}$$

For $\sigma_2 = \sigma_3 = \sigma$ and $\sigma_1 = \sigma + 2\pi$ we get:

$$\{y_1(\sigma + 2\pi), y_2(\sigma), y_3(\sigma)\} \cong \frac{2H}{k}. \qquad (37)$$

Conclusion

After two T-dualizations we had that Q flux theory is commutative.

Closed string noncommuatativity and nonassociativity are consequence of the fact that Kalb-Ramond field is coordinate dependent.

Parameters of noncommutativity and nonassociativity are proportional to the field strength H.

In ordinary space, coordinate dependent background is a sufficient condition for closed string noncommutativity.

THANKS
FOR YOUR ATTENTION