Lie 2-groups and Quantum Gravity

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2-group

- A 2-group is a 2-category with one object where all the 1-morphisms and all the 2-morphisms are invertible. Also called a "categorical group", i.e. monoidal category with the invertible product and the invertible morphisms.
- A strict 2-group is equivalent to a *crossed module*: a pair of groups (G, H) with a group action $\triangleright : G \times H \to H$ and a homomorphism $\partial : H \to G$ such that

$$\partial(g \triangleright h) = g(\partial h)g^{-1}\,, \quad (\partial h) \triangleright h' = hh'h^{-1}\,.$$

- The morphisms are the elements of G, while the 2-morphisms are the elements of the semi-direct product group $G \times_s H$.
- Poincaré 2-group: $G = SO(3,1)$, $H = \mathbf{R}^4$,

$$g \triangleright h = h' \Leftrightarrow \Lambda(g)v(h) = v(h')$$

and $\partial h = 1_G$.

2-connection

- A crossed module of Lie groups (G, H) is called a Lie 2-group. If M is a manifold, then a 2-connection on M is a pair of forms (A, β) on M such that A is a one-form taking values in $\mathbf{g}$, and β is a two-form taking values in $\mathbf{h}$. The gauge transformations are

$$A \to g^{-1}(A + d)\, g\,, \quad \beta \to g^{-1} \rhd \beta$$

for $g : M \to G$ (thin gauge transformations) and

$$A \to A + \partial\epsilon\,, \quad \beta \to \beta + d\epsilon + A \wedge^{\rhd} \epsilon + \epsilon \wedge \epsilon$$

where ϵ is a one-form from $\mathbf{h}$ (fat gauge transformations) and

$$A \wedge^{\rhd} \epsilon = A^I \wedge \epsilon^\alpha \, \Delta_{I\alpha}^{\beta} T_\beta\,.$$

- Δ are the structure constants defined by the group action $\rhd$ for the corresponding Lie algebras. Hence $X_I \rhd T_\alpha = \Delta_{I\alpha}^{\beta} T_\beta$, where X is a basis for $\mathbf{g}$ and T is a basis for $\mathbf{h}$.

2-connection

- In the Poincaré 2-group case

$$A(x) = \omega^{ab}(x)\, J_{ab}\,, \quad \beta(x) = \beta^a(x)\, P_a\,,$$

where J are the Lorentz group generators and P are the translation generators.

- Infinitesimal thin gauge transformations

$$\delta_\lambda \omega^{ab} = d\lambda^{ab} + \omega^{[a}_{c}\, \lambda^{b]c}\,, \quad \delta_\lambda \beta^a = \lambda^a_c\, \beta^c\,.$$

- Infinitesimal fat gauge transformations

$$\delta_\epsilon \omega = 0\,, \quad \delta_\epsilon \beta^a = d\epsilon^a + \omega^a_c \wedge \epsilon^c\,.$$

2-curvature

- The curvature for a 2-connection (A, β) is a pair of a 2-form $\mathcal{F} \in \mathbf{g}$ and a 3-form $\mathcal{G} \in \mathbf{h}$, given by

$$\mathcal{F} = dA + A \wedge A - \partial\beta \,, \quad \mathcal{G} = d\beta + A \wedge^{\triangleright} \beta \,.$$

- Gauge transformations

$$\mathcal{F} \to g^{-1} \mathcal{F} g \,, \quad \mathcal{G} \to g^{-1} \triangleright \mathcal{G} + \mathcal{F} \wedge^{\triangleright} \epsilon \,.$$

- In the Poincaré 2-group case, we have

$$\mathcal{F}^{ab} \equiv R^{ab} = d\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb}$$

$$\mathcal{G}^{a} \equiv G^{a} = \nabla\beta^{a} = d\beta^{a} + \omega^{a}{}_{c} \wedge \beta^{c} \,,$$

so that R^{ab} is the usual spin-connection curvature. The $\partial\beta$ term does not appear in R^{ab} beacuse $\partial\beta = 0$ for the Poincaré 2-group.

BFCG theory

- The dynamics of flat 2-connections for the Poincaré 2-group is given by the BFCG action

$$S = \int_M \left(B_{ab} \wedge R^{ab} + e_a \wedge G^a \right)$$

where B^{ab} is a 2-form and e_a is a one form (a tetrad).

- The Lagrange multipliers B and e transform under the usual (thin) gauge transformations as

$$B \to g^{-1} Bg \,, \quad e \to g \triangleright e \,,$$

while the fat gauge transformations are given by

$$B_{ab} \to B_{ab} + e_{[a} \wedge \epsilon_{b]} \,, \quad e_a \to e_a \,.$$

BFCG theory

- If a constraint

$$B_{ab} = \epsilon_{abcd}\, e^c \wedge e^d \,,$$

is imposed in the BFCG action, one obtains a theory which is equivalent to the Einstein-Cartan formulation of General Relativity

$$S_{EC} = \int_M \epsilon^{abcd}\, e_a \wedge e_b \wedge R_{cd} \,.$$

- More precisely, the action

$$S_{2PC} = \int_M \left[B_{ab} \wedge R^{ab} + e_a \wedge G^a - \phi^{ab} \wedge \left(B_{ab} - \epsilon_{abcd} e^c \wedge e^d \right) \right] \,.$$

is dynamically equivalent to S_{EC}.

Canonical formulation

- *Dirac procedure*: Given an action for variables $Q : \mathbf{R} \to \mathbf{R}^n$
$$S = \int_a^b L(Q, \dot{Q})\, dt \,,$$
where $\dot{Q} = dQ/dt$, then S is dynamically equivalent to
$$S_D = \int_a^b dt \left[P_k \dot{Q}^k - H_0(P,Q) - \lambda^a \, G_a(P,Q) - \mu^\alpha \, \theta_\alpha(P,Q) \right] \,.$$
- The first-class constraintsl satisfy
$$\{G_a, G_b\}_D = f_{ab}{}^c(P,Q)\, G_c \,, \quad \{G_a, H_0\}_D = h_a^b(P,Q)\, G_b \,,$$
where the *Dirac bracket* is given by
$$\{X, Y\}_D = \{X, Y\} - \{X, \theta_\alpha\}\Delta^{\alpha\beta}\{\theta_\beta, Y\} \,,$$
$\Delta = \{\theta, \theta\}^{-1}$ and
$$\{X, Y\} = \frac{\partial X}{\partial Q^k}\frac{\partial Y}{\partial P_k} - \frac{\partial X}{\partial P_k}\frac{\partial Y}{\partial Q^k} \,.$$

Canonical formulation

- If $L(Q,\dot{Q}) = p_i \dot{q}^i - \lambda^c G_c(p,q)$ and

$$\{G_c, G_d\}^* = f_{cd}{}^e(p,q)\, G_e\,,$$

where $\{,\}^*$ is the (p,q) Poisson bracket, then S is a gauge-fixed form of S_D where the second-class constraints have been eliminated and some of the phase-space coordinates have been set to zero.
- In the Poincare BFCG case, let $M = \Sigma \times \mathbf{R}$, and

$$X_{\mu\cdots}\, Y^{\mu\cdots} = X_{0\cdots}\, Y^{0\cdots} + X_{i\cdots}\, Y^{i\cdots}\,,$$

so that

$$\mathcal{L} = \pi^i_{ab}\dot{\omega}^{ab}_i + \Pi^{ij}_a \dot{\beta}^a_{ij} - \lambda_1 \mathcal{C}_1 - \lambda_2 \mathcal{C}_2 - \Lambda_1 \mathcal{G}_1 - \Lambda_2 \mathcal{G}_2\,,$$

where

$$\mathcal{C}^i_{1ab} = \frac{1}{2}\epsilon^{ijk}\, R_{ab\,jk}\,, \quad \mathcal{C}^a_2 = \frac{1}{2}\epsilon^{ijk}\nabla_i \beta^a_{jk}\,,$$

$$\mathcal{G}_{1ab} = \nabla_i \pi^i_{ab} - \beta_{[a|ij}\, \Pi^{ij}_{b]}\,, \quad \mathcal{G}^{ai}_2 = \nabla_j \Pi^{aji}\,,$$

Canonical formulation

and

$$\pi^{ab\,i} = \frac{1}{2}\,\epsilon^{ijk} B^{ab}_{jk}\,, \quad \Pi^{a\,ij} = -\frac{1}{2}\,\epsilon^{ijk}\,e^a_k\,.$$

- Constraint algebra

$$\{C_I(x), C_J(y)\} = f_{IJ}^K\, C_K(x)\,\delta(x-y)\,,$$

where $C_I \in \{C^a, C^{ab}, C^a_{ij}, C^{ab}_{ij}\} = \{\mathcal{C}^a_2, \mathcal{G}^{ab}_1, \mathcal{G}^a_{2ij}, \mathcal{C}^{ab}_{1ij}\}$.

- Poincare BFCG is dynamically the same as the Poincare BF theory

$$\int_M \left(B^{ab}\wedge R_{ab} + e^a \wedge \nabla\beta_a\right) = \int_M \left(B^{ab}\wedge R_{ab} + \nabla e^a \wedge \beta_a\right)$$

$$= \int_M \left(B^{ab}\wedge R_{ab} + \beta^a \wedge T_a\right)\,.$$

Canonical formulation

- Canonical formulation for BF Poincare:

$$A_i^I = (\omega_i^{ab}, e_i^a)\,, \quad E_I^i = (\pi^i{}_{ab}, p_a^i)\,,$$

and the constraint algebra is the same as in the 2-Poincare case, since

$$(C^I, C_{ij}^I) = (C^a, C^{ab}, C_{ij}^a, C_{ij}^{ab})\,.$$

- Relation to the canonical formulation of the 2-Poincare theory: $(e,p) \to (\beta, \Pi)$ such that

$$\beta_{ij}^a = \varepsilon_{ijk}\, p^{ak}\,, \quad \Pi_a^{ij} = -\epsilon^{ijk}\, e_{ak}\,.$$

Canonical quantization

- $(p,q) \in \mathbf{R}^n \times Q_n \to (\hat{p},\hat{q})$ acting on $L_2(Q_n)$ such that
$$\hat{p}_k\,\Psi(q) = i\,\frac{\partial\Psi(q)}{\partial q^k}\,,\quad \hat{q}_k\,\Psi(q) = q_k\,\Psi(q)\,.$$
- Constrained CQ: solve $\hat{C}_I\Phi(q) = 0$ such that $\Phi(q) \in L_2(Q^*)$ and $Q^* \subset Q_n$.
- In the BF case Q^* is the space of flat connections on Σ modulo gauge transformations, i.e. $Q^* = \mathcal{M}(\Sigma)$. Hence
$$Q^*_{PBF} = \mathcal{M}_{ISO(3,1)}(\Sigma) = VB(\mathcal{M}_{SO(3,1)})\,,$$
where the fibers are solutions of $de + \omega\wedge e = 0$ and $\omega \in \mathcal{M}_{SO(3,1)}$.
- Because of the dynamical equivalence
$$Q^*_{2PF} \simeq Q^*_{PBF}\,.$$

Loop quantization

- Instead of (A_i^l, E_l^i) use $(Hol_\gamma(A), \Phi_\sigma(B))$ to quantize, where $\gamma = \partial\sigma$ and $B_{ij} = \epsilon_{ijk} E^k$.
- Represent the flux-holonomy algebra in the spin-network basis

$$W_{\hat\gamma}(A) = Tr\left(\prod_{v\in\gamma} C^{(\iota_v)} \prod_{l\in\gamma} D^{(\Lambda_l)}(A)\right) \equiv \langle A|\hat\gamma\rangle\,,$$

  where $\hat\gamma = (\gamma, \Lambda, \iota)$ denotes a spin network associated to a closed graph γ.
- When A is a flat connection, than W is invariant under a homotopy of the graph γ, so that we can label the spin-network wavefunctions by combinatorial (abstract) graphs γ.

Loop quantization

- By requiring that $W_{\hat{\gamma}}(A)$ form a basis in $\mathcal{H}$, we obtain

$$|\Psi\rangle = \int DA\,|A\rangle\langle A|\Psi\rangle = \sum_{\hat{\gamma}} |\hat{\gamma}\rangle\langle\hat{\gamma}|\Psi\rangle$$

where

$$\langle\hat{\gamma}|\Psi\rangle = \int DA\,\langle\hat{\gamma}|A\rangle\langle A|\Psi\rangle = \int DA\,W^*_{\hat{\gamma}}(A)\,\Psi(A)$$

is the loop transform.

2-holonomy quantization

- How to generalize the spin-network basis $W_{\hat{\gamma}}(A)$ to a spin-foam basis $W_{\hat{\Gamma}}(A,\beta)$ where $\hat{\Gamma}=(\Gamma,L,\Lambda,\iota)$?
- Conjecture

$$W_{\hat{\Gamma}}(\omega_l,\beta_f)=Tr\left(\prod_{v\in\Gamma}C^{(\iota_v)}\prod_{l\in\Gamma}D^{(\Lambda_l)}(\omega)\prod_{f\in\Gamma}D^{(L_f)}(\omega,\beta)\right)$$

where

$$D^{(L_f)}(\omega,\beta)=D^{(L_f)}(g_{l(f)}\triangleright h_f)\,,$$

and $f\in\partial p$ such that

$$h_p=\prod_{f\in\partial p}g_{l(f)}\triangleright h_f\ ,$$

is a 2-holonomy.

Conclusions

- In the 2-Poincare case one can also use

$$W_{\hat{\Gamma}}(\omega_l, \beta_f) \sim W_{\hat{\gamma}}(\omega_l, e_l) \sim \tilde{W}_{\hat{\gamma}}(\omega_l, \beta_\Delta)\,,$$

  where $\tilde{W}_{\hat{\gamma}}$ is the Fourier transform of $W_{\hat{\gamma}}$. This may give clues about a Peter-Weil theorem for 2-groups.
- $SU(2)$ spin-network basis can be generalized to a spin-foam basis (edge lengths and face areas) for the 3d Euclidean 2-group.
- Canonical quantization in the spin-foam basis requires a canonical formulation of GR with $(A_i^\alpha, \tilde{\beta}_{ij}^\alpha)$ variables, $\alpha = 1, 2, 3$ (Euclidean 2-group 2-connection).
- Finding $(A_i^\alpha, \tilde{\beta}_{ij}^\alpha)$ variables requires a gauge fixing to

$$\{\,(\omega_i^\alpha, \pi_\alpha^i)\,,\,(\beta_{ij}^\alpha, \Pi_\alpha^{ij})\,\}\ \textit{or}\ \{\,(\omega_i^\alpha, \pi_\alpha^i)\,,\,(e_\alpha^i, p_i^\alpha)\,\}$$

  canonical variables.

Conclusions

- The results of the canonical analysis of the BFCG formulation of GR (A. Miković, M.Oliveira and M. Vojinović; arxiv:1807.06354) imply

$$L = \int_\Sigma d^3x \, [\, p_X \dot{X} - \lambda^\alpha G_\alpha(p_X, X) - n^i \mathcal{D}_i(p_X, X) - N\mathcal{H}(p_X, X)$$

$$- \sum_{k=1}^{18} \mu_k \theta_k(p_X, X) \,] \,,$$

where $X = (\omega_i^\alpha, e_i^\alpha)$, G are the Gauss constraints, $\mathcal{D}$ are the 3-diffeomorphism constraints, $\mathcal{H}$ is the Hamiltonian constraint and θ are the second-class constraints.

- In the RPS quantization one can then use the following Eucledean 2-group 2-connection

$$(A_i^\alpha, \tilde{\beta}_{ij}^\alpha) = (\omega_i^\alpha, \epsilon_{ijk} e^{k\alpha}) \,.$$

References

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