Courant and Roytenberg bracket and their relations to the current algebra in bosonic string theory

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Overview

Action

$$S[x] = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}[x] + \epsilon^{\alpha\beta} B_{\mu\nu}[x]\right] \partial_\alpha x^\mu \partial_\beta x^\nu$$

- bosonic string in the coordinate dependant background
- string is propagating in D=26 dimensions
- worldsheet is parametrized with one time-like ($\xi_0 = \tau$) and one space-like ($\xi_1 = \sigma$) parameter

$$S[x] = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu}[x] \partial_- x^\nu, \quad \Pi_{\pm\mu\nu}[x] = B_{\mu\nu}[x] \pm \frac{1}{2} G_{\mu\nu}[x]$$

$$\Theta^{\mu\nu}_\pm = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}$$

T-duality

- T-duality is an equivalence of two seemingly different physical theories in a way that all observable quantities in one theory are identified with quantities in its dual theory
- T-duality connects different superstring theories: two flavors of heterotic string theory (SO(32) and $E_8 \times E_8$) and theories of type IIa and IIb
- T-duality is used in a construction of equivalent Calabi-Yau manifolds.

Example: Closed string in the background with one dimension compactified on the radius R

- $M^2 = \frac{K^2}{R^2} + W^2 \frac{R^2}{\alpha'^2}$
- Spectrum doesn't change under the simultaneous transformations $K \leftrightarrow W$ and $R \leftrightarrow \frac{\alpha'}{R}$
- Momenta in one theory are winding numbers in its T-dual theory and vice versa

Construction of the T-dual theory of sigma model

- Buscher procedure
  - at least one Abelian isometry is required
  - isometry is gauged by introducing the gauge fields
  - gauge fields should not carry the additional degrees of freedom
- Generalized Buscher procedure
  - constructed for certain coordinate-dependant background fields
  - invariant coordinate as the line integral of the covariant derivatives of the original coordinates

Action:

$$^\star S[y] = \kappa \int d^2\xi \partial_+ y_\mu {}^\star\Pi_+^{\mu\nu}(\Delta V(y))\partial_- y_\nu$$

Currents in the original theory

- Hamiltonian as a function of some currents $j_{\pm\mu}$:

$$H = \frac{1}{4\kappa} j_{-\mu} (G^{-1})^{\mu\nu} j_{-\nu} + \frac{1}{4\kappa} j_{+\mu} (G^{-1})^{\mu\nu} j_{+\nu}$$

$$j_{\pm\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu} x'^\nu$$

- Currents can be rewritten in the following form:

$$j_{\pm\mu} = i_\mu \pm \kappa G_{\mu\nu} x'^\nu, \quad i_\mu = \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu$$

Currents in the T-dual theory

- Hamiltonian as a function of currents ${}^\star k_\pm^\mu$ in dual theory:

$$
{}^\star H = \frac{1}{4\kappa}{}^\star k_-^\mu G^E_{\mu\nu}{}^\star k_-^\nu + \frac{1}{4\kappa}{}^\star k_+^\mu G^E_{\mu\nu}{}^\star k_+^\nu
$$

$$
{}^\star k_\pm^\mu = {}^\star\pi^\mu + 2{}^\star\Pi_\pm^{\mu\nu} y'_\nu
$$

- Currents can be rewritten in the following form:

$$
{}^\star k_\pm^\mu = {}^\star l^\mu \pm (G_E^{-1})^{\mu\nu} y'_\nu, \quad {}^\star l^\mu = {}^\star\pi^\mu + 2{}^\star B^{\mu\nu} y'_\nu
$$

- They satisfy the following relations:

$$
{}^\star k_\pm^\mu \cong l^\mu \pm \kappa x'^\mu + \kappa\theta_\mp^{\mu\nu}\pi_\nu = l^\mu \pm \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}\pi_\nu, \qquad (1)
$$

$$
{}^\star l_\pm^\mu \cong \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu
$$

General currents

The most general currents:

$$J_{C(u,\alpha)} = u^\mu(x) i_\mu + \alpha_\mu(x) x'^\mu$$
$$J_{R(u,\alpha)} = u^\mu(x) \pi_\mu + \alpha_\mu(x) l^\mu$$

Currents $j_{\pm\mu}$ and $k_\pm^\mu$ can be obtained with following transformations:

$$\alpha_\mu = \pm\kappa G_{\mu\nu} u^\nu, \; J_{C(u,\alpha)} \to u^\mu j_{\pm\mu}$$
$$u^\mu = \pm\frac{1}{\kappa}(G_E^{-1})^{\mu\nu}\alpha_\nu, \; J_{R(u,\alpha)} \to \alpha_\mu k_\pm^\mu$$

Current algebra

- Poisson brackets between the currents:

$$\{i_\mu(\sigma), i_\nu(\sigma')\} = -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma - \sigma')$$
$$\{l^\mu(\sigma), l^\nu(\sigma')\} = -\kappa Q_\rho{}^{\mu\nu} l^\rho \delta(\sigma - \sigma') - \kappa^2 R^{\mu\nu\rho} \pi_\rho \delta(\sigma - \sigma')$$

- Structure constants are given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$
$$Q_\rho{}^{\mu\nu} = \partial_\rho \theta^{\mu\nu}$$
$$R^{\mu\nu\rho} = \theta^{\mu\sigma} \partial_\sigma \theta^{\nu\rho} + \theta^{\nu\sigma} \partial_\sigma \theta^{\rho\mu} + \theta^{\rho\sigma} \partial_\sigma \theta^{\mu\nu}$$

Current algebra

Algebra between the most general currents:

$$\{J_{C(u,\alpha)}(\sigma), J_{C(v,\beta)}(\sigma')\} = J_{C(w,\gamma)}\delta(\sigma-\sigma') + \Big((u,\alpha),(v,\beta)\Big)\delta'(\sigma-\sigma')$$

$$\begin{aligned} w^\mu &= (v^\nu \partial_\nu u^\mu - u^\nu \partial_\nu v^\mu) \\ \gamma_\rho &= -2B_{\mu\nu\rho} u^\mu v^\nu + u^\mu(\partial_\rho \beta_\mu - \partial_\mu \beta_\rho) + (\alpha_\mu \partial_\rho + \partial_\mu \alpha_\rho) v^\mu \\ \Big((u,\alpha),(v,\beta)\Big) &= u^\mu \beta_\mu + \alpha^\mu v_\mu \end{aligned}$$

$$\{J_{R(u,\alpha)}(\sigma), J_{R(v,\beta)}(\sigma')\} = J_{R(w,\gamma)}\delta(\sigma-\sigma') + \Big((u,\alpha),(v,\beta)\Big)\delta'(\sigma-\sigma')$$

$$\begin{aligned} w^\mu = {} & \partial_\nu u^\mu v^\nu - \partial_\nu v^\mu u^\nu - \kappa \alpha_\nu \theta^{\nu\rho} \partial_\rho v^\mu - \kappa \alpha_\nu \partial_\rho v^\nu \theta^{\nu\rho} + \kappa \alpha_\nu v^\rho \partial_\rho \theta^{\nu\mu} + \\ & \kappa \theta^{\nu\rho} \partial_\rho u^\mu \beta_\nu - \kappa \partial_\nu \theta^{\rho\mu} \beta_\rho u^\nu - \kappa \partial_\nu \beta_\rho u^\rho \theta^{\nu\mu} - \kappa^2 R^{\mu\nu\rho} \alpha_\nu \beta_\rho \\ \gamma_\mu = {} & \partial_\nu \alpha_\mu v^\nu + \alpha_\nu \partial_\mu v^\nu + \kappa \theta^{\nu\rho} \partial_\rho \alpha_\mu \beta_\nu - \kappa \theta^{\nu\rho} \partial_\rho \beta_\mu \alpha_\nu + \\ & u^\nu (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) - \kappa Q_\mu{}^{\nu\rho} \alpha_\nu \beta_\rho \end{aligned}$$

Charges

- Charges are defined as usual:

$$Q_{C(u,\alpha)} = \int d\sigma J_{C(u,\alpha)}(\sigma), \quad Q_{R(u,\alpha)} = \int d\sigma J_{R(u,\alpha)}(\sigma)$$

- They satisfy following algebra:

$$\{Q_{C(u,\alpha)}, Q_{C(v,\beta)}\} = -Q_{C[(u,\alpha),(v,\beta)]_C}$$
$$\{Q_{R(u,\alpha)}, Q_{R(v,\beta)}\} = -Q_{R[(u,\alpha),(v,\beta)]_R}$$

Courant bracket

- Courant bracket is a generalization of the Lie bracket.
- Lie bracket is the operation on the tangent bundle, while Courant bracket is the operation on the direct sum of the tangent bundle and the vector bundle of 1-forms.

$$[u+\alpha, v+\beta]_C = [u,v]_L + L_u\beta - L_v\alpha - \frac{1}{2}d(i_u\beta - i_v\alpha) + H(u,v,.)$$

- Courant bracket does not satisfy the Jacobi identity. The Jacobiator of the Courant bracket is an exact form.

In our theory:

$$\begin{aligned}[u+\alpha, v+\beta]_C = &(v^\nu\partial_\nu u^\mu - u^\nu\partial_\nu v^\mu) + u^\mu(\partial_\rho\beta_\mu - \partial_\mu\beta_\rho) \\ &- v^\mu(\partial_\rho\alpha_\mu - \partial_\mu\alpha_\rho)v^\mu - 2B_{\mu\nu\rho}u^\mu v^\nu\end{aligned}$$

Roytenberg bracket

- Roytenberg bracket is a generalization of Courant bracket, in a way that it includes a bi-vector $\Pi = \frac{1}{2}\Pi^{\mu\nu}\partial_\mu\partial_\nu$ as well.

$$
\begin{aligned}
[u+\alpha, v+\beta]_R =& [u,v]_L + L_u\beta - L_v\alpha - \frac{1}{2}d(i_u\beta - i_v\alpha) + H(u,v,.) - \\
& H\Pi(u,v) + \Pi H(\alpha, v, .) - \Pi H(\beta, u, .) \\
& (L_v\alpha - L_u\beta + \frac{1}{2}d(i_u\beta - i_v\alpha))\Pi + \\
& \Lambda^2\Pi H(\alpha, ., v) - \Lambda^2\Pi H(\beta, ., u) - [\alpha,\beta]_\Pi + \\
& \Lambda^2\Pi H(\alpha,\beta,.) - [v, \alpha\Pi]_L + [u, \beta\Pi]_L + \\
& \left(\frac{1}{2}[\Pi,\Pi]_S - \Lambda^3\Pi H\right)(\alpha,\beta,.)
\end{aligned}
$$

Questions?