p-ADIC STRING THEORY

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Gravity and String Theory:
New Ideas for Unsolved Problems III
(In honor of Prof. Branislav Sazdović's retirement)
7 – 9.09, 2018
Zlatibor, SERBIA

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p-adic mathematical physics (1987)

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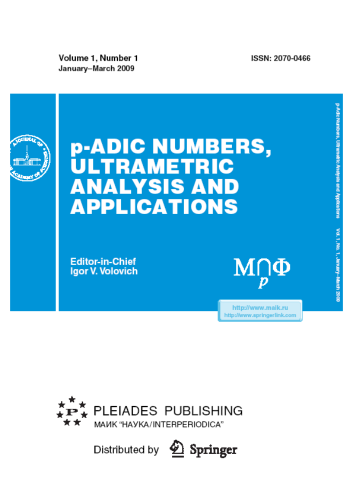
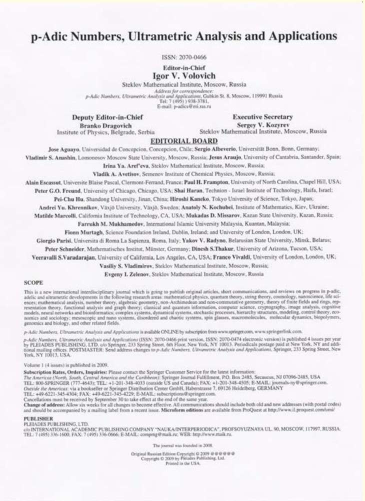
Some review references

- L. Brekke and P.G.O. Freund, *p-adic numbers in physics*, Phys. Rep. **233**, 1–66 (1993).
- V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, *p-Adic Analysis and Mathematical Physics* (World Scientific, Singapore, 1994).
- B. Dragovich, A.Yu. Khrennikov, S.V. Kozyrev and I.V. Volovich, *On p-adic mathematical physics*, *p*-Adic Numbers Ultrametric Anal. Appl. 1 (1), 1–17 (2009). arXiv:0904.4205v1[math-ph].
- B. Dragovich, A. Yu. Khrennikov, S. V. Kozyrev, I. V. Volovich and E. I. Zelenov, *p-Adic mathematical physics: the first 30 years*, *p*-Adic Numbers Ultrametric Anal. Appl. 9 (2), 87–121 (2017). arXiv:1705.04758 [math-ph].

1. Introduction: international journal on *p*-adics





p-Adic Numbers, Ultrametric Analysis and Applications

ISSN: 2070-0466

Editor-in-Chief
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p-Adic Numbers, Ultrametric Analysis and Applications is available ONLINE by subscription from www.springer.com, www.springerlink.com.

p-Adic Numbers, Ultrametric Analysis and Applications (ISSN: 2070-0466 print version, ISSN: 2070-0474 electronic version) is published 4 issues per year by PLEIADES PUBLISHING, LTD. c/o Springer, 233 Spring Street, 6th Floor, New York, NY 10013. Periodicals postage paid at New York, NY and additional mailing offices. POSTMASTER: Send address changes to *p-Adic Numbers, Ultrametric Analysis and Applications*, Springer, 233 Spring Street, New York, NY 10013, USA.

Volume 1 (4 issues) is published in 2009.

Subscription Rates, Orders, Inquiries: Please contact the Springer Customer Service for the latest information:
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PUBLISHER
PLEIADES PUBLISHING, LTD.
c/o INTERNATIONAL ACADEMIC PUBLISHING COMPANY "NAUKA/INTERPERIODICA", PROFSOYUZNAYA UL. 90, MOSCOW, 117997, RUSSIA.
TEL: 7 (495) 336-1600; FAX: 7 (495) 336-0666; E-MAIL: compmg@maik.ru; WEB: http://www.maik.ru.

The journal was founded in 2008.

1. Introduction: international conferences

- Moscow 2003, Belgrade 2005, Moscow 2007, Grodno 2009, Bielefeld 2013, Belgrade 2015, Mexico 2017, Portugal 2019

2nd International Conference on p-Adic Mathematical Physics
Belgrade, 2005

2. *p*-Adic numbers, adeles and their functions

- Ostrowski theorem: Any nontrivial norm on $\mathbb{Q}$ is equivalent to usual absolute value or to *p*-adic norm, where *p* is any prime number $(p = 2, 3, 5, 7, 11, \dots)$.
- *p*-adic norm of $x \in \mathbb{Q}$: $\quad |x|_p = |p^\nu \frac{a}{b}|_p = p^{-\nu}\,, \quad \nu \in \mathbb{Z}$.
- $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ and $\mathbb{R}$.
- Completion of $\mathbb{Q}$ with respect to *p*-adic distance gives the field $\mathbb{Q}_p$ of *p*-adic numbers, in analogous way to construction of the field $\mathbb{R}$ of real numbers.

$$
\begin{tikzcd}
 & & & \mathbb{R} \arrow[r, dash] & \mathbb{C} \\
\mathbb{N} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \mathbb{Q} \arrow[ur, dash, "|\cdot|_\infty"] \arrow[dr, dash, "|\cdot|_p"'] & & \\
 & & & \mathbb{Q}_p \arrow[r, dash] & \mathbb{C}_p
\end{tikzcd}
$$

2. p-Adic numbers, adeles and their functions

Examples

p-adic norms of $x = \frac{21}{20} = \frac{3 \cdot 7}{2^2 \cdot 5}$

- $|\frac{21}{20}|_2 = 4$
- $|\frac{21}{20}|_3 = \frac{1}{3}$
- $|\frac{21}{20}|_5 = 5$
- $|\frac{21}{20}|_7 = \frac{1}{7}$
- $|\frac{21}{20}|_p = 1$ if $p \geq 11$.

p-adic expansion of -1

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + (p-1)p^3 + \ldots$$

2. p-Adic numbers, adeles and their functions

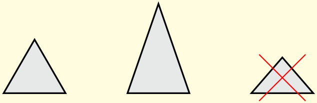
- p-Adic numbers invented by Kurt Hensel (1861-1941) in 1897.
- Any p-adic number ($x \in \mathbb{Q}_p$) has a unique canonical representation

$$x = p^\nu \sum_{n=0}^{+\infty} x_n p^n \,, \ \nu \in \mathbb{Z} \,, \ x_n \in \{0, \, 1, \, \cdots \, , \, p-1\}$$

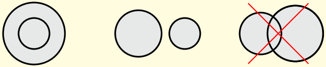
$$|x_n p^{(\nu+n)}|_p = p^{-(\nu+n)}, \quad |x+y|_p \leq max\{|x|_p, |y|_p\}$$

2. p-Adic numbers, adeles and their functions

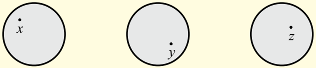
- p-adic distances $d_p(x, y) = |x - y|_p$ are non-Archimedean (ultrametric): $d_p(x, y) \leq max\{d_p(x, z), d_p(z, y)\}$
- open ball of radius r and centre a:
- closed ball of radius r and centre a:
- balls are simultaneously closed and open



All triangles are isosceles.



There is no partial intersection of balls.



Any point of a ball is its center.

2. p-Adic numbers, adeles and their functions

- Real and p-adic numbers are unified by adeles. An adele α is an infinite sequence

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \cdots, \alpha_p, \cdots), \quad \alpha_\infty \in \mathbb{R}, \ \alpha_p \in \mathbb{Q}_p$$

where for all but a finite set S of primes p one has that $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, i.e. p-adic integers.

- Space of adeles

$$\mathbb{A} = \bigcup_S A(\mathcal{S}), \quad A(S) = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p .$$

C. Chevalley (1909 - 1984)

A. Weil (1906 - 1998)

2. *p*-Adic numbers, adeles and their functions

- Recall standard theoretical models

$$\textit{classical models}: \ \mathbb{R} \to \mathbb{R}$$

$$\textit{quantum models}: \ \mathbb{R} \to \mathbb{C} \to \mathbb{R}$$

- *p*-Adic modeling

$$\textit{classical models}: \ \mathbb{Q}_p \to \mathbb{R}$$

$$\textit{quantum models}: \mathbb{Q}_p \to \mathbb{C} \to \mathbb{R}$$

- *p*-Adic side of some phenomena may emerge at a deeper level.

2. *p*-Adic numbers, adeles and their functions

Some connections of *p*-adic and real norms of rational numbers

$$|x|_\infty \times \prod_{p\in\mathbb{P}} |x|_p = 1\,, \text{ if } x \in \mathbb{Q}^\times$$

$$\chi_\infty(x) \times \prod_{p\in\mathbb{P}} \chi_p(x) = 1\,, \text{ if } x \in \mathbb{Q}$$

$$\chi_\infty(x) = \exp(-2\pi i x), \qquad \chi_p(x) = \exp 2\pi i \{x\}_p$$

$$\prod_{p\in\mathbb{A}} \Omega(|x|_p) = \begin{cases} 1, & x \in \mathbb{Z}, \\ 0, & x \in \mathbb{Q}\setminus\mathbb{Z} \end{cases}$$

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \le 1, \\ 0, & |x|_p > 1 \end{cases}$$

3. p-Adic strings

Volovich, Freund, Witten, Vladimirov, Arefeva, B. D., ...

String amplitudes (1987):

- standard crossing symmetric Veneziano amplitude

$$A_\infty(a,b) = g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1}\, |1-x|_\infty^{b-1}\, d_\infty x$$

$$= g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}$$

- p-adic crossing symmetric Veneziano amplitude

$$A_p(a,b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1}\, |1-x|_p^{b-1}\, d_p x$$

$$= g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}}$$

where $a = -s/2 - 1$ and $a, b, c \in \mathbb{C}$ and $a + b + c = 1$.

3. *p*-Adic strings

- *p*-Adic strings are strings which world-sheet is *p*-adic.
- Freund-Witten (adelic) product formula for ordinary and *p*-adic strings

$$A(a,b) = A_\infty(a,b)\prod_p A_p(a,b) = g_\infty^2 \prod_p g_p^2 = const.$$

- Ordinary and *p*-adic strings are at the equal footing
- Amplitude for real string $A_\infty(a,b)$, which is a special function, can be presented as product of inverse *p*-adic amplitudes, which are elementary functions.

3. p-Adic strings – Effective Lagrangian

Freund, Witten, Frampton, Okada, ...

- There is an effective field description of scalar open and closed p-adic strings. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher ones at the tree-level.
- The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\,\frac{p^2}{p-1}\Big[-\frac{1}{2}\,\varphi\, p^{-\frac{\Box}{2m_p^2}}\,\varphi + \frac{1}{p+1}\,\varphi^{p+1}\Big]$$

where $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and metric with signature $(- + \dots +)$.

3. p-Adic strings

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\,\frac{p^2}{p-1}\Big[-\frac{1}{2}\,\varphi\, p^{-\frac{\Box}{2m_p^2}}\,\varphi + \frac{1}{p+1}\,\varphi^{p+1}\Big]$$

- Lagrangian with p-adic world-sheet B.D.

$$\mathcal{L}_p = \frac{m^D}{g^2}\,\frac{p^2}{p-1}\Big[\frac{1}{2}\,\varphi \int_{\mathbb{R}} \Big(\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} \chi_p(u)|u|_p^{\frac{k^2}{2m^2}}\,du\Big)\tilde{\varphi}(k)\,\chi_\infty(kx)\,d^4k$$

$$+\frac{1}{p+1}\,\varphi^{p+1}\Big]$$

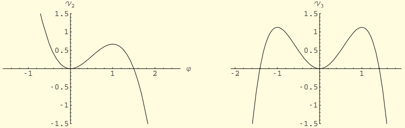
$$\int_{|x|_p>1} \chi_p(u)|u|_p^s du = -p^s$$

- The corresponding potential $\mathcal{V}(\varphi)$ for this Lagrangian is $\mathcal{V}_p(\varphi) = -\mathcal{L}_p(\Box = 0)$, which the explicit form is

$$\mathcal{V}_p(\varphi) = \frac{m^D}{g^2}\Big[\frac{1}{2}\frac{p^2}{p-1}\varphi^2 - \frac{p^2}{p^2-1}\varphi^{p+1}\Big].$$

  It has local minimum $\mathcal{V}_p(0) = 0$. If $p \neq 2$ there are two local maxima at $\varphi = \pm 1$ and there is one local maximum $\varphi = +1$ when $p = 2$.

- The 2-adic string potential $\mathcal{V}_2(\varphi)$ (on the left) and 3-adic potential $\mathcal{V}_3(\varphi)$ (on the right) of the figure.



3. p-Adic strings – equation of motion

- The equation of motion is

$$p^{-\frac{\Box}{2m_p^2}}\,\varphi = \varphi^p$$

- It has trivial solutions $\varphi = 0$ and $\varphi = 1$, and $\varphi = -1$ for $p \neq 2$.
- There are also solutions in any direction x_i and time t

$$\varphi(x_i) = p^{\frac{1}{2(p-1)}}\,\exp\Big(-\frac{p-1}{2\,p\ln p} m^2\,x_i^2\Big)$$

$$\varphi(t) = p^{\frac{1}{2(p-1)}}\,\exp\Big(\frac{p-1}{2\,p\ln p} m^2 t^2\Big)$$

3. p-Adic strings

- There is also D-dimensional solution

$$\varphi(x) = p^{\frac{D}{2(p-1)}} \exp\Big(-\frac{p-1}{2\,p\ln p} m^2 x^2\Big)\,, \quad x^2 = -t^2 + \sum_{i=1}^{D-1} x_i^2$$

- The above solutions can be obtained using identity

$$e^{A\,\partial_t^2}\, e^{B\,t^2} = \frac{1}{\sqrt{1-4AB}}\, e^{\frac{B\,t^2}{1-4AB}}, \quad 1-4AB > 0.$$

- p-Adic realization of tachyon condensation

3. p-Adic strings – Lagrangian for p-adic string sector

B.D.
Consider

$$L = \sum_{n=1}^{+\infty} C_n \, \mathcal{L}_n = m^D \sum_{n=1}^{+\infty} \frac{C_n}{g_n^2} \frac{n^2}{n-1} \Big[-\frac{1}{2} \, \phi \, n^{-\frac{\Box}{2m^2}} \, \phi + \frac{1}{n+1} \, \phi^{n+1} \Big]$$

where $\frac{C_n}{g_n^2} \frac{n^2}{n-1} = (-1)^{n-1}$. Take into account

$$\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s}) \, \zeta(s), \quad s = \sigma + i\tau \, , \quad \sigma > 0$$

which has analytic continuation to the entire complex s plane without singularities.

3. p-Adic strings – Lagrangian for p-adic string sector

- The corresponding Lagrangian is

$$L = m^D \Big[-\frac{1}{2}\phi\Big(1 - 2^{1-\frac{\Box}{2m^2}}\Big)\zeta\Big(\frac{\Box}{2m^2}\Big)\phi + \phi - \frac{1}{2}\log(1+\phi)^2\Big].$$

- The potential is

$$V(\phi) = -L(\Box = 0) = m^D\Big[\frac{1}{4}\phi^2 - \phi + \frac{1}{2}\log(1+\phi)^2\Big],$$

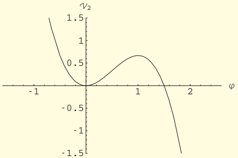
which has one local maximum $V(0) = 0$ and one local minimum at $\phi = 1$. It is singular at $\phi = -1$.

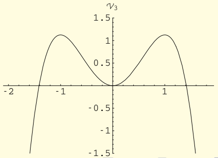
- The equation of motion is

$$\Big(1 - 2^{1-\frac{\Box}{2m^2}}\Big)\zeta\Big(\frac{\Box}{2m^2}\Big)\phi = \frac{\phi}{1+\phi}, \quad \phi = 0.$$

4. Concluding remarks

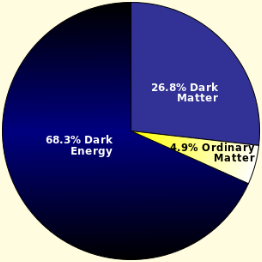
- Taking limit $p = 1 + \varepsilon \to 1$ in effective Lagrangian, one obtains
$$L_1(\varphi) = \frac{m^D}{g^2}\Big[\frac{1}{2}\varphi\frac{\Box}{m^2}\varphi + \frac{\varphi^2}{2}(\ln\varphi^2 - 1)\Big]$$
which corresponds to the ordinary bosonic string in the boundary string field theory.
- p-Adic tachyon condensation
- p-Adic inflation
- p-Adic AdS/CFT correspondence





4. Concluding remarks

- If Einstein general theory of relativity is theory of gravity for the Universe as a whole, then there is only about 5% of ordinary matter, about 27% of dark matter and about 68% of dark energy.
- Conjecture: Dark matter and dark energy have *p*-adic origin.
- Modified gravity may be nonlocal, e.g. $R \to R + RF(\Box)R$.



4. Concluding remarks

- p-Adic numbers are not results of measurements, but can be useful in physical models.
- p-Adic strings are strings with world-sheet labeled by p-adic numbers.
- If there exist ordinary strings then there should exist also p-adic strings.
- p-Adic string theory is simpler than ordinary string theory, and is useful for ordinary string theory.
- p-Adic strings may play significant role in description of dark side of the universe.