T-duality of an open string with mixed boundary conditions

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Outline

- Modified open string action
- Open string boundary conditions (Neumann and Dirichlet boundary conditions)
- Dirac procedure
- Solving the constraints
- Generalized T-dualization procedure
- Solving the constraints of T-dual theory
- Effective theories compared

Bosonic string action

- The action

$$S[x] = \kappa \int_{\Sigma} d^2\xi \sqrt{-g} \Big[\Big(\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \Big) \cdot \partial_\alpha x^\mu \partial_\beta x^\nu + \frac{1}{4\pi\kappa} \Phi(x) R^{(2)} \Big]$$

- The background
  - The metric field $G_{\mu\nu} = G_{\nu\mu}$
  - The Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$
  - The dilaton field Φ

Bosonic string action in a conformal gauge

- $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$
- Action

$$S[x] = \kappa \int_\Sigma d^2\xi \; \partial_+ x^\mu \, \Pi_{+\mu\nu}(x) \, \partial_- x^\nu, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma$$

- Background field composition

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x)$$

Closed or open string

- The general coordinate transformations

$$\delta_\xi G_{\mu\nu} = -2(D_\mu \xi_\nu + D_\nu \xi_\mu), \quad D_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho_{\mu\nu} \xi_\rho$$

$$\delta_\xi B_{\mu\nu} = -2\xi^\rho B_{\rho\mu\nu} + 2(\partial_\mu b_\nu - \partial_\nu b_\mu), \quad b_\mu = B_{\mu\nu} \xi^\nu$$

- The local gauge transformations

$$\delta_\Lambda G_{\mu\nu} = 0$$

$$\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$

- Variations of action

$$\delta S = 2\sqrt{2}\kappa \int d\tau \Big[\Pi_{-\mu\nu}\, \xi^\nu \partial_- x^\mu + \Pi_{+\mu\nu}\, \xi^\nu \partial_+ x^\mu \Big] \Big|_{\sigma=0}^{\sigma=\pi}$$

$$\delta S = 2\kappa \int d\tau\, \Lambda_\mu \dot{x}^\mu \Big|_{\sigma=0}^{\sigma=\pi}$$

Open string

- Surface term

$$S_{\partial\Sigma} = 2\int d\tau \Big[\kappa A_\mu(x)\dot{x}^\mu - \bar{A}_\mu(x)(G^{-1})^{\mu\nu}\gamma_\nu^{(0)}\Big]\Big|_{\sigma=0}^{\sigma=\pi}$$

- The vector fields transform as

$$\delta_\Lambda A_\mu = -\Lambda_\mu, \qquad \delta_\xi \bar{A}_\mu = -\xi_\mu$$

The mixed boundary conditions

$$\gamma_\mu^{(0)} \delta x^\mu \Big|_{\sigma=0,\pi} = 0$$

$$\gamma_\mu^{(0)} = G_{\mu\nu} x'^\nu - 2B_{\mu\nu} \dot{x}^\nu$$

- The Neumann conditions for x^a, $a = 0, 1, \dots, p$
- The Dirichlet conditions for x^i, $i = p+1, \dots, D-1$

$$S_{\partial\Sigma} = 2 \int d\tau \Big[\kappa A_a^N(x) \dot{x}^a - A_i^D(x) (G^{-1})^{ij} \gamma_j^{(0)} \Big] \Big|_{\sigma=0}^{\sigma=\pi}$$

$$\gamma_j^{(0)} = \Pi_{+j\mu} \partial_- x^\mu + \Pi_{-j\mu} \partial_+ x^\mu$$

Hamiltonian

$$\mathcal{H}_C = T_- - T_+$$

- Energy momentum tensor

$$T_\pm = \mp \frac{1}{4\kappa}(G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu}$$

- Currents

$$j_{\pm\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu}(x) x'^\nu$$

- Momentum

$$\pi_\mu = -2\kappa B_{\mu\nu} x'^\nu + \kappa G_{\mu\nu} \dot{x}^\nu$$

The boundary conditions canonical form

$$\gamma_a^{(0)} = \frac{1}{\kappa}\Pi_{+a\mu}(G^{-1})^{\mu\nu} j_{-\nu} + \frac{1}{\kappa}\Pi_{-a\mu}(G^{-1})^{\mu\nu} j_{+\nu}$$

$$\gamma^i = \dot{x}^i = \frac{1}{2\kappa}(G^{-1})^{i\mu}\left(j_{+\mu} + j_{-\mu}\right)$$

The Dirac procedure

- The zeroth constrains are the boundary conditions.
- Infinite number of constraints

$$\gamma_a^1 = \{H, \gamma_a^0\}, \qquad \gamma_a^n = \{H, \gamma_a^{n-1}\}$$

$$\gamma_1^i = \{H, \gamma^i\}, \qquad \gamma_n^i = \{H, \gamma_{n-1}^i\}$$

- Space parameter dependent constraints

$$\Gamma_a^N(\sigma) = \sum_{n\geq 0} \frac{\sigma^n}{n!}\gamma_a^n, \qquad \Gamma_D^i(\sigma) = \sum_{n\geq 0} \frac{\sigma^n}{n!}\gamma_n^i$$

Current algebra

$$\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar\sigma)\} = \pm 2\kappa \Gamma_{\mp\mu,\nu\rho} x'^{\rho}(\sigma)\delta(\sigma-\bar\sigma) \pm 2\kappa G_{\mu\nu}(x(\sigma))\delta'(\sigma-\bar\sigma)$$

$$\{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar\sigma)\} = \pm 2\kappa \Gamma_{\mp\rho,\mu\nu} x'^{\rho}(\sigma)\delta(\sigma-\bar\sigma)$$

- The generalized connection

$$\Gamma_{\pm\mu,\nu\rho} = \Gamma_{\mu,\nu\rho} \pm B_{\mu\nu\rho}$$

  - The Christoffel symbol

$$\Gamma_{\mu,\nu\rho} = \frac{1}{2}(\partial_\nu G_{\rho\mu} + \partial_\rho G_{\mu\nu} - \partial_\mu G_{\nu\rho})$$

  - The field strength of the field $B_{\mu\nu}$

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

The constant background

$$G_{\mu\nu} = \begin{pmatrix} G_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} B_{ab} & 0 \\ 0 & B_{ij} \end{pmatrix}.$$

- The current algebra

$$\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar\sigma)\} = \pm\, 2\kappa\, G_{\mu\nu}\, \delta'(\sigma-\bar\sigma), \quad \{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar\sigma)\} = 0$$

- The first constraint $\{H, j_{\pm\mu}\} = \mp\, j'^{\mu}_{\pm}$

The σ dependent constraints

- $$\Gamma_a^N(\sigma) = \frac{1}{\kappa}\sum_{n\geq 0}\frac{\sigma^n}{n!}\left[\Pi_{+a\mu}(G^{-1})^{\mu\nu}(-1)^n j_{-\nu}^{(n)} + \Pi_{-a\mu}(G^{-1})^{\mu\nu} j_{+\nu}^{(n)}\right]$$

  $$\Gamma_D^i(\sigma) = \frac{1}{2\kappa}\sum_{n\geq 0}\frac{\sigma^n}{n!}(G^{-1})^{i\mu}\left[j_{+\mu}^{(n)} + (-1)^n j_{-\mu}^{(n)}\right]$$

- The final form

  $$\Gamma_a^N = \frac{1}{\kappa}\left[\Pi_{+a\mu}(G^{-1})^{\mu\nu} j_{-\nu}(-\sigma) + \Pi_{-a\mu}(G^{-1})^{\mu\nu} j_{+\nu}(\sigma)\right]$$

  $$\Gamma_D^i = \frac{1}{2\kappa}(G^{-1})^{i\mu}\left[j_{+\mu}(\sigma) + j_{-\mu}(-\sigma)\right]$$

The explicit form of the constraints

- Odd and even variables

$$x^\mu = q^\mu + \bar{q}^\mu, \qquad \pi_\mu = p_\mu + \bar{p}_\mu$$

- The constraints

$$\Gamma^N_a = \frac{1}{\kappa}\Big[2B_a{}^\nu p_\nu - \bar{p}_a - \kappa (G_E)_{a\rho}\bar{q}'^\rho\Big]$$

$$\Gamma^i_D = \frac{1}{\kappa}(G^{-1})^{i\mu}\Big[p_\mu + \kappa G_{\mu\nu} q'^\nu + 2\kappa B_{\mu\nu}\bar{q}'^\nu\Big]$$

- The solution

$$x'^\mu = q'^a - \theta^{ab} p_b + \bar{q}'^i$$

$$\pi_\mu = p_a - 2\kappa B_{ij}\bar{q}'^j + \bar{p}_i$$

Generalized Buscher procedure

- Old steps (applicable to backgrounds which do not depend on the coordinates which one T-dualizes):
  - Localize the global symmetry $\delta x^\mu = \lambda^\mu = const$
  - Introduce the gauge fields v_α^μ
  - Substitute the ordinary derivatives with the covariant derivatives

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v_\alpha^\mu$$

  - Impose the transformation law for the gauge fields

$$\delta v_\alpha^\mu = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau, \sigma))$$

- New step (enables T-dualization of every coordinate):
  - Substitute the coordinate x^μ by the invariant coordinate

$$\Delta x_{inv}^\mu \equiv \int_P d\xi^\alpha \, D_\alpha x^\mu = x^\mu - x^\mu(\xi_0) + \Delta V^\mu,$$

here

$$\Delta V^\mu \equiv \int_P d\xi^\alpha v_\alpha^\mu.$$

Generalized Buscher procedure

- Old step:
  - Require the equivalence with the initial theory
  - Field strength

$$F^{\mu}_{\alpha\beta} \equiv \partial_\alpha v^{\mu}_{\beta} - \partial_\beta v^{\mu}_{\alpha}$$

    must be zero
  - Add the Lagrange multiplier y_μ term in the Lagrangian
- Result:
  - Gauge invariant action

$$S_{inv} = \kappa \int d^2\xi \Big[D_+ x^\mu \Pi_{+\mu\nu}(\Delta x_{inv}) D_- x^\nu + \frac{1}{2}(v^{\mu}_{+}\partial_- y_\mu - v^{\mu}_{-}\partial_+ y_\mu)\Big]$$

  - Fix the gauge $x^\mu(\xi) = x^\mu(\xi_0)$
  - Gauge fixed action

$$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[v^{\mu}_{+}\Pi_{+\mu\nu}(\Delta V) v^{\nu}_{-} + \frac{1}{2}(v^{\mu}_{+}\partial_- y_\mu - v^{\mu}_{-}\partial_+ y_\mu)\Big]$$

T-duality

$$S[x] = \kappa \int_\Sigma d^2\xi \ \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu$$

$$\cong$$

$$^\star S[y] = \frac{\kappa^2}{2} \int d^2\xi \ \partial_+ y_\mu \Theta_-^{\mu\nu}[\Delta V^{(0)}(y)] \partial_- y_\nu$$

- T-dual coordinate transformation law

$$\partial_\pm x^\mu \cong -\kappa \, \Theta_\pm^{\mu\nu}[\Delta V^{(0)}] \partial_\pm y_\nu \mp 2\kappa \Theta_{0\pm}^{\mu\nu} \beta_\nu^\mp [V^{(0)}]$$

T-dual theory

- T-dual action

$$^\star S_0 = \frac{\kappa^2}{2} \int d\xi^2 \partial_+ y_\mu \Theta_-^{\mu\nu} \partial_- y_\nu$$

- T-dual background filed composition

$$\Theta_\pm^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1} B G^{-1})^{\mu\nu} = \begin{pmatrix} \Theta_\pm^{ab} & 0 \\ 0 & \Theta_\pm^{ij} \end{pmatrix},$$

- the inverse of the initial metric

$$(G^{-1})^{\mu\nu} = \begin{pmatrix} (G^{-1})^{ab} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix}$$

- the effective metric

$$(G_E)_{\mu\nu} = \begin{pmatrix} (G_E)_{ab} & 0 \\ 0 & (G_E)_{ij} \end{pmatrix}$$

T-dual surface term

$$^\star S_{\partial\Sigma} = -2\kappa \int d\tau \Big[A^N_a(V)^\star\gamma^{(0)a} - \frac{1}{\kappa} A^D_i(V)(G^{-1})^{ij}\dot{y}_j \Big]\Big|_{\partial\Sigma}$$

- The boundary condition

$$^\star\gamma^{(0)a} = \frac{\kappa}{2}\Big[\Theta^{a\mu}_{-}\partial_{-}y_\mu + \Theta^{a\mu}_{+}\partial_{+}y_\mu\Big]$$

- The supposed form of the surface term

$$2\kappa \int d\tau \Big[{}^\star A^i_N \dot{y}_i - \frac{1}{\kappa}{}^\star A^a_D({}^\star G^{-1})_{ab}{}^\star\gamma^{(0)b} \Big]\Big|_{\partial\Sigma}$$

- Dual vector field on the boundary

$$^\star A^i_N = \frac{1}{\kappa}(G^{-1})^{ij}A^D_j$$

$$^\star A^a_D = \kappa(G^{-1}_E)^{ab}A^N_b$$

The T-dual Hamiltonian

- Dual momentum

$$^\star\pi^\mu = -\kappa^2\theta^{\mu\nu}y'_\nu + \frac{\kappa}{2}(G_E^{-1})^{\mu\nu}\dot{y}_\nu$$

- Dual currents

$$^\star j_\pm^\mu = {}^\star\pi^\mu + 2\kappa{}^\star\Pi_\pm^{\mu\nu}y'_\nu$$

- Dual energy momentum tensor

$$^\star T_\pm = \mp\frac{1}{4\kappa}({}^\star G^{-1})_{\mu\nu}{}^\star j_\pm^\mu {}^\star j_\pm^\nu$$

- Hamiltonian

$$\mathcal{H}_C = {}^\star T_- - {}^\star T_+$$

The dual constraints

- The Neumann condition

$$^\star\gamma^{(0)i} = \frac{\kappa}{2}\Big[\Theta_-^{i\mu}\partial_- y_\mu + \Theta_+^{i\mu}\partial_+ y_\mu\Big] = \frac{1}{2}\Big[\Theta_-^{i\mu} G^E_{\mu\nu}\, {}^\star j_-^\nu + \Theta_+^{i\mu} G^E_{\mu\nu}\, {}^\star j_+^\nu\Big]$$

- The Dirichlet condition

$$^\star\gamma_a = \dot{y}_a = \frac{1}{2\kappa} G^E_{a\mu}\Big({}^\star j_+^\mu + {}^\star j_-^\mu\Big),$$

- Parameter dependent constraints

$$^\star\Gamma^i_N = \frac{1}{2}\Big[\Theta_-^{i\mu} G^E_{\mu\nu}\, {}^\star j_-^\nu(-\sigma) + \Theta_+^{i\mu} G^E_{\mu\nu}\, {}^\star j_+^\nu(\sigma)\Big]$$

$$^\star\Gamma^D_a = \frac{1}{2\kappa} G^E_{a\mu}\Big({}^\star j_+^\mu(\sigma) + {}^\star j_-^\mu(-\sigma)\Big)$$

Separating dual variables and currents

- odd and even coordinates and momenta

$$y_\mu = k_\mu + \bar{k}_\mu$$

$$^\star\pi^\mu = {}^\star p^\mu + {}^\star\bar{p}^\mu$$

- odd and even currents

$$^\star j^\mu_\pm(\sigma) = {}^\star p^\mu(\sigma) + {}^\star\bar{p}^\mu(\sigma) + \kappa^2\Theta^{\mu\nu}_\mp(k'_\nu(\sigma) + \bar{k}'_\nu(\sigma))$$

$$^\star j^\mu_\pm(-\sigma) = {}^\star p^\mu(\sigma) - {}^\star\bar{p}^\mu(\sigma) + \kappa^2\Theta^{\mu\nu}_\mp(-k'_\nu(\sigma) + \bar{k}'_\nu(\sigma))$$

- The sigma dependent constraints

$$^\star\Gamma^i_N = -\frac{2}{\kappa}B^i{}_\mu{}^\star p^\mu - \frac{1}{\kappa}{}^\star\bar{p}^i - (G^{-1})^{i\mu}\bar{k}'_\mu$$

$$^\star\Gamma^D_a = \frac{1}{\kappa}G^E_{a\mu}({}^\star p^\mu + \kappa^2\theta^{\mu\nu}_0\bar{k}'_\nu + \kappa(g_E^{-1})^{\mu\nu}k'_\nu)$$

The solution

- The constraints

$$^\star\Gamma^i_N\Big|_{\sigma=0} = 0, \qquad ^\star\Gamma^D_a\Big|_{\sigma=0} = 0$$

- The solution

$$y'_\mu = k'_i - \frac{2}{\kappa} B_{ij}{}^\star p^j + \bar{k}'_a$$
$$^\star\pi^\mu = {}^\star p^i + {}^\star\bar{p}^a - \kappa^2\theta^{ab}\bar{k}'_b$$

Noncommutativity

- initial theory

$$x'^{\mu} = q'^{a} - \theta^{ab} p_b + \bar{q}'^{i}$$

$$\pi_{\mu} = p_a - 2\kappa B_{ij} \bar{q}'^{j} + \bar{p}_i$$

- completely T-dualized theory

$$y'_{\mu} = k'_i - \frac{2}{\kappa} B_{ij} {}^{\star}p^{j} + \bar{k}'_a$$

$${}^{\star}\pi^{\mu} =^{\star} p^{i} + {}^{\star}\bar{p}^{a} - \kappa^2 \theta^{ab} \bar{k}'_b$$

The effective hamiltonians

- for initial theory

$$\mathcal{H}_{eff} = \frac{\kappa}{2}\, q'^a G^E_{ab} q'^b + \frac{1}{2\kappa} p_a (G_E^{-1})^{ab} p_b + \frac{\kappa}{2}\, \bar{q}'^i G_{ij} \bar{q}'^j + \frac{1}{2\kappa} \bar{p}_i (G^{-1})^{ij} \bar{p}_b$$

- for dual theory

$$^\star\mathcal{H}_{eff} = \frac{\kappa}{2}\, \bar{k}'_a (G_E^{-1})^{ab} \bar{k}'_b + \frac{1}{2\kappa} {}^\star\bar{p}^a (G_E)_{ab} {}^\star\bar{p}^b + \frac{\kappa}{2}\, k'_i (G^{-1})^{ij} k'_j + \frac{1}{2\kappa} {}^\star p^i G_{ij} {}^\star p^j$$

The effective lagrangians

- for initial theory

$$\mathcal{L}_{eff} = \frac{\kappa}{2}(G_E)_{ab}(\dot{q}^a\dot{q}^b - q'^a q'^b) + \frac{\kappa}{2}G_{ij}(\dot{\bar{q}}^i\dot{\bar{q}}^j - \bar{q}'^i\bar{q}'^j)$$

- for dual theory

$$^\star\mathcal{L}_{eff} = \frac{\kappa}{2}(G_E^{-1})^{ab}(\dot{\bar{k}}^a\dot{\bar{k}}^b - \bar{k}'^a\bar{k}'^b) + \frac{\kappa}{2}(G^{-1})^{ij}(\dot{k}^i\dot{k}^j - k'^i k'^j)$$

What is a symmetry transformation of fields?

- The change in fields which does not change the classical action.
- World-sheet action
- Space-time fields on a world-sheet
- The corresponding conformal field theories
- Change in energy-momentum tensor

Change of the energy-momentum tensor

- Hamiltonian

$$\mathcal{H}_C = T_- - T_+$$

- Virasoro algebras

$$\begin{aligned}\Big[\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\pm(\varphi(\bar\sigma))\Big] &= i\hbar\Big[\hat{T}_\pm(\varphi(\sigma)) + \hat{T}_\pm(\varphi(\bar\sigma))\Big]\delta'(\sigma-\bar\sigma) \\ \Big[\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\mp(\varphi(\bar\sigma))\Big] &= 0\end{aligned}$$

- Similarity transformation $T_\pm \to e^{-i\Gamma} T_\pm e^{i\Gamma}$
-

$$\delta T_\pm(\varphi) = -i\Big[\Gamma, T(\varphi)\Big]$$

Closed string algebra

- Suppose the background fields undergo a small change in value $\Pi_{\pm\mu\nu} \to \Pi_{\pm\mu\nu} + \delta\Pi_{\pm\mu\nu}$
- The current changes by

$$\delta j_{\pm\mu} = 2\kappa\delta\Pi_{\pm\mu\nu}(x)x'^{\nu}$$

- The energy-momentum tensor $T_{\pm} = \mp\frac{1}{4\kappa}(G^{-1})^{\mu\nu}j_{\pm\mu}j_{\pm\nu}$ changes by

$$\delta T_{\pm} = \frac{1}{2\kappa}\delta\Pi_{\pm\mu\nu}j^{\mu}_{\pm}j^{\nu}_{\mp}$$

- The standard Poisson brackets between the coordinates and the momenta

$$\{x^{\mu}(\sigma), \pi_{\nu}(\bar{\sigma})\} = \delta^{\mu}_{\nu}\delta(\sigma - \bar{\sigma})$$

Closed string symmetry generator

The symmetry generator $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$

$$\mathcal{G}_\pm = \int d\sigma\, \Lambda^\mu_\pm(x(\sigma)) j_{\pm\mu}(\sigma)$$

- $\mathcal{G}_\pm$ satisfy: $\{T_\pm(\sigma), \mathcal{G}_\pm(\bar\sigma)\} = \pm\frac{1}{2\kappa}\Big(D_{\mp\nu}\Lambda^\mu_\pm\Big) j^\nu_\mp j_{\pm\mu}$

$$\{T_\pm(\sigma), \mathcal{G}_\mp(\bar\sigma)\} = \pm\frac{1}{2\kappa}\Big(D_{\pm\nu}\Lambda^\mu_\mp\Big) j^\nu_\pm j_{\mp\mu}$$

  - The covariant derivatives $D_{\pm\mu}\Lambda^\nu = \partial_\mu\Lambda^\nu + \Gamma^\nu_{\pm\rho\mu}\Lambda^\rho = D_\mu\Lambda^\nu \pm B^\nu{}_{\rho\mu}\Lambda^\rho$
- One demands

$$\delta T_\pm = \{\mathcal{G}, T_\pm\} = \frac{1}{2\kappa}\delta\Pi_{\pm\mu\nu} j^\mu_\pm j^\nu_\mp$$

Outlook

- Find the complete form of the effective theories
- Find the symmetry transformations of the background fields and their generators
- Include more complicate background into consideration
- Consider the partial T-dualization as well