(Non)integrability and the bound on chaos in topological black hole geometries

Mihailo Čubrović
Institute of Physics Belgrade

Gravity and String Theory III, Zlatibor, Serbia, 2018.

Outline

- Integrability in nature and in string theory
- Topological black holes
- (Non)integrability – analytical and numerical
- Bound on chaos?

Outline

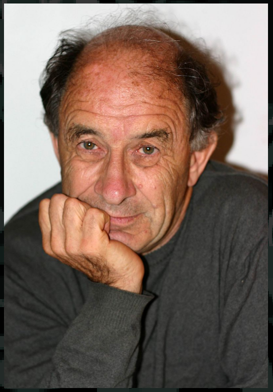
- Integrability in nature and in string theory
- Topological black holes
- (Non)integrability – analytical and numerical
- Bound on chaos?

Integrability

- Everybody knows: an N degree-of-freedom integrable system has N independent integrals of motion
- In detail: several different definitions
- Not only a mathematical curiosity: crucial for deeper understanding
- Quantum integrability – even tougher problem
- In this talk classical only!

Kolmogorov-Arnol'd-Moser

A. N. Kolmogorov



V. I. Arnol'd

- KAM theory – geometry of the phase space
- Action-angle variables and invariant tori

Coordinates & momenta

(p, q)

Canonical transformation →

Actions=integrals of motion

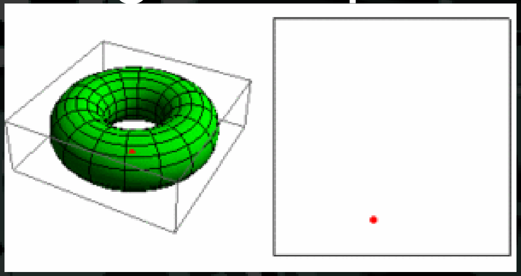
Angles periodic

$(I, \varphi): \quad I = \text{const.}, \qquad \varphi \sim \sin \omega t$

- No algorithmic way to find action-angle variables

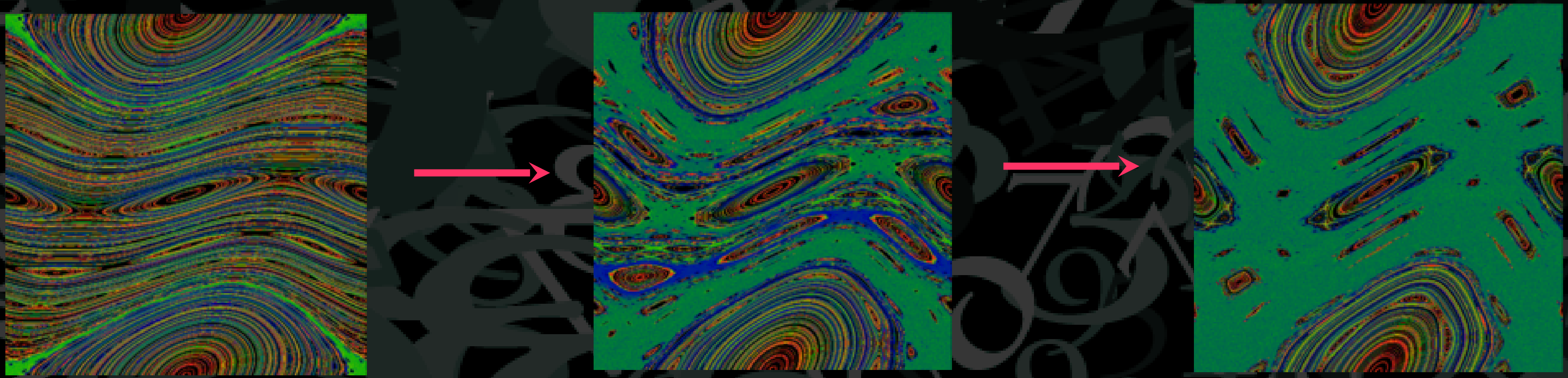
Kolmogorov-Arnol'd-Moser

- Integrable: phase space foliated by tori



Periodic motion on the torus (here rotation; libration also possible)

- Nonintegrable with perturbation ϵ : progressive destruction of invariant tori but some still remain until we reach ϵ_{crit}



$\epsilon < \epsilon_{crit}$ $\epsilon \approx \epsilon_{crit}$ $\epsilon > \epsilon_{crit}$

Kolmogorov-Arnol'd-Moser

- Some orbits stable for all times, but some others can be arbitrarily chaotic
- Effective Langevin equation for actions in the vicinity of a torus:

$$\dot{I} = -\epsilon\, \partial K_1 / \partial \varphi \rightarrow \langle \dot{I} \rangle = \epsilon\, F_1(I)\, \eta(t)$$

- How relevant this "Arnol'd diffusion" is depends on timescales:

  - solar system $t_{diff} \sim 10^{10} t_0 \quad 10^{10}\,\text{years}$

  - confined plasmas $t_{diff} \sim 10^{9} t_0 \quad 10\,\text{days}$

Differential Galois theory

- Galois theory in an algebraic field with a differential operator (Leibniz rule)
- Consider functions from a differential field F with constant subfield C and simple extension E

Differential Galois theory

- Galois theory in an algebraic field with a differential operator (Leibniz rule)
- Consider functions from a differential field F with constant subfield C and simple extension E
- Can an ODE be integrated by quadratures? <-> is there such an E that it has the same C as F but is closed to inverses of differential operations?
- Extends the intuition that integrals of rational functions are polynomials possibly multipled by logs
- Can be implemented algorithmically with some limitations – Kovacic algorithm

The foundation – Liouville theorem

- Is a Hamiltonian H on the phase space M integrable?

↓

- Find an invariant submanifold P.

↓

- Project the Hamiltonian EOMs X on P: $X|_P$

↓

- Find variational equations $\delta X|_P$ in a tangen plane to P

↓

- Now H is integrable if the largest connected subgroup of the Galois group is Abelian

Integrability in string theory

- Relevant for quantization, integrability in gauge theories (including but not limited to AdS/CFT)
- Particles (geodesics) and strings: Arutyunov, Nekrasov, Tseytlin, Lunin... 2000s, 2010s
- D-brane stacks: one or two parallel stacks integrable (Chervonyi&Lunin 2014), base needs to be of the form:

$$ds_b^2 = dr_1^2 + r_1^2 d\,\Omega_{d_1}^2 + dr_2^2 + r_2^2 d\,\Omega_{d_2}^2$$

- Stepanchuk&Tseytlin 2013: integrability established for $\mathrm{AdS}_p \times S^q$ (and for flat space); brane configurations that interpolate between them nonintegrable

Integrability in string theory

- Simple geometries explored by Basu & Pando Zayas (2010s)
- Planar and AdS Schwarzschild, planar and AdS RN (nonextremal), $AdS_p \times SE^q$ (Sasaki-Einstein manifold), AdS soliton nonintegrable
- Extremal black holes should be integrable if the bound on chaos conjecture is to be believed: bound proportional to temperature, no chaos around T=0 horizon

Outline

- Integrability in nature and in string theory
- Topological black holes
- (Non)integrability – analytical and numerical
- Bound on chaos?

Construction from AdS

- Old story, apparently not very popular these days
- Event horizon – surface of higher genus
- M. Banados, R. B. Mann, S. Holst, P. Peldan and others

Construction from AdS

- Old story, apparently not very popular these days
- Event horizon – surface of higher genus
- M. Banados, R. B. Mann, S. Holst, P. Peldan and others
- Start from AdS_{N+1} and identify the points in the R_M Minkowski subspace ($M \leq N$ connected by some discrete subgroup of the SO(M-1,1) isometry
- To avoid the closed timelike curves first restrict to the subspace $x_0^2 - x_i x^i = R_+^2 / L^2 > 0$ – gives a compact subspace of negative curvature:

$$ds_M^2 = d\varphi_1^2 + \sinh^2\varphi_1 d\varphi_2^2 + \sinh^2\varphi_1 \sinh^2\varphi_2 d\varphi_3^2 + \ldots$$

Constructiong from AdS

- To avoid the closed timelike curves restrict to the subspace AdS_{N+1} ($M \leq N$) – gives a compact subspace of negative curvature:

$$ds_M^2 = d\varphi_1^2 + \sinh^2\varphi_1\, d\varphi_2^2 + \sinh^2\varphi_1 \sinh^2\varphi_2\, d\varphi_3^2 + \ldots$$

- The remaining coordinates define a BH horizon by a change of variables: $(x_M, x_{M+1}, \ldots x_{N+1}) \rightarrow (t, R, \theta_1, \theta_2, \ldots \theta_{N-M-1})$
- The metric:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 \left(d\varphi_1^2 + \sinh^2\varphi_1\, d\varphi_2^2 + \ldots \right) + \frac{L^4}{R_+^2} \cosh^2\left(\frac{R_+}{L} t \right) \left(d\theta_1^2 + \sinh^2\theta_1\, d\theta_2^2 + \ldots \right)$$

- BH can be charged by picking the appropriate $f(r)$

Higher genus horizons

- Identify now the points related by an isometry from SO(M-1,1): $ds_M^2 \rightarrow dH_g^2$ - surface of genus $g \in N$
- Toric BH (g=1): $ds_M^2 = d\varphi_1^2 + d\varphi_2^2 + d\varphi_3^2 + \ldots$
- Spherical BH (g=0): $ds_M^2 = d\varphi_1^2 + \sin^2\varphi_1 d\varphi_2^2 + \sin^2\varphi_1 \sin^2\varphi_2 d\varphi_3^2 + \ldots$
- The metric:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 dH_g^2 + \frac{L^4}{R_+^2} \cosh^2\left(\frac{R_+}{L} t\right)\left(d\theta_1^2 + \sinh^2\theta_1 d\theta_2^2 + \ldots\right)$$

- Solution of Einstein equations in the vacuum for negative constant dilaton

Identification of points

- Toric BH (g=1): $ds_M^2 = d\varphi_1^2 + d\varphi_2^2 + d\varphi_3^2 + \ldots$ - infinite hyperplane if no identification is made
- Requirements: sum of angles $\geq 2\pi$ to avoid conical singularities; 4g sides needed: for g=1 -> square -> wrapping (identification) yields a torus

Identification of points

- Toric BH (g=1): $ds_M^2 = d\varphi_1^2 + d\varphi_2^2 + d\varphi_3^2 + \ldots$ - infinite hyperplane if no identification is made
- Requirements: sum of angles $\geq 2\pi$ to avoid conical singularities; 4g sides needed: for g=1 -> square -> wrapping (identification) yields a torus
- Hyperbolic BH: $ds_M^2 = d\varphi_1^2 + \sinh^2\varphi_1 \, d\varphi_2^2 + \sinh^2\varphi_1 \sinh^2\varphi_2 \, d\varphi_3^2 + \ldots$
- Again need sum of angles $\geq 2\pi$ and 4g sides but sums of angles on a pseudosphere have a lesser sum than on a plane -> minimal g=2

Topological BH formation

- Collapes of presureless dust – but need to start from the AdS space with identifications (Mann&Smith 1997)

Topological BH formation

- Collapes of presureless dust – but need to start from the AdS space with identifications (Mann&Smith 1997)
- Cosmological C-metric – dynamical, more realistic (Mann 1997, Kaloper 1997)

Topological BH formation

- Collapes of presureless dust – but need to start from the AdS space with identifications (Mann&Smith 1997)
- Cosmological C-metric – dynamical, more realistic (Mann 1997, Kaloper 1997)
- BH with fermionic hair (possible in AdS) with a Berry phase (Čubrović 2018) – purely formal but can be related to cond-mat systems

$$S_\Psi = \int d^{N+1}x\sqrt{-g}\,\bar{\Psi}\left(D_a\Gamma^a - m\right)\Psi + \oint d^N x\sqrt{h}\left(\bar{\Psi}_- e^{\frac{i\varphi\Gamma^\varphi}{2}}\Psi_+ - \bar{\Psi}_-\Psi_+\right)$$

$$T_{ab} = \langle \bar{\Psi} D_a \Gamma_b \Psi \rangle$$

Feed this into the Einstein equations

Surface term introduces Berry phase

- Backreaction by fermions introduces topological horizon

Outline

- Integrability in nature and in string theory
- Topological black holes
- (Non)integrability – analytical and numerical
- Bound on chaos?

Closed string in TBH background

- Polyakov action:

$$S = \frac{1}{2\pi\alpha'} \int d\tau\, d\sigma \left(\eta_{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \epsilon_{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\mu \right)$$

(circled term $\epsilon_{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\mu$ → 0)

- Gauge $h_{ab} = \eta_{ab}$ -> Virasoro constraints:

$$\eta_{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu = 0, \quad \epsilon_{ab} \partial_a X^\mu \partial_b X^\nu = 0$$

Closed string in TBH background

- Polyakov action:

$$S = \frac{1}{2\pi\alpha'} \int d\tau\, d\sigma \left(\eta_{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \epsilon_{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\mu \right)$$

(annotation: the $\epsilon_{ab} B_{\mu\nu}$ term is circled, with an arrow to 0)

- Gauge $h_{ab} = \eta_{ab}$ -> Virasoro constraints:

$$\eta_{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu = 0, \quad \epsilon_{ab} \partial_a X^\mu \partial_b X^\nu = 0$$

- Ansatz:
  - point-like dynamics ~ just chaos, no turbulence -> nontrivial dependence only on τ
  - three DOF -> $R(\tau),\ T(\tau)$ + either $\Theta_1(\tau)$ or $\Phi_1(\tau)$

Closed string in TBH background

- Dynamical Θ_1 (1) or dynamical Φ_1 (2) with winding along Θ_2 (a) or Φ_2 (b)
- Six cases:
- (1a) $T(\tau), R(\tau), \Theta_1(\tau);\ \Theta_2(\sigma)=n\sigma,\ \ M=2, N=4$
- (1b) $T(\tau), R(\tau), \Theta_1(\tau);\ \Phi_1(\sigma)=p\sigma,\ \ M=2, N=3$
- (1ab) $T(\tau), R(\tau), \Theta_1(\tau);\ \Theta_2(\sigma)=n\sigma, \Phi_1(\sigma)=p\sigma,\ \ M=2, N=4$
- (2a) $T(\tau), R(\tau), \Phi_1(\tau);\ \Theta_1(\sigma)=n\sigma,\ \ M=2, N=3$
- (2b) $T(\tau), R(\tau), \Phi_1(\tau);\ \Phi_2(\sigma)=p\sigma,\ \ M=2, N=3$
- (2ab) $T(\tau), R(\tau), \Phi_1(\tau);\ \Theta_1(\sigma)=n\sigma, \Phi_2(\sigma)=p\sigma,\ \ M=2, N=4$

Expectations

- Planar and AdS non-extremal black branes and black holes nonintegrable. What could go right with TBHs?
- (i) just a single equilibrium point instead of infinity along Θ coordinates
- (ii) horizons with negative mass term in $f(R)$ possible, might influence the possibility to express the coefficients of the linearized equations as rational functions

Integrable TBH

- Consistent (3+1)d truncation from the (4+1)d case (2b):

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 \left(d\varphi_1^2 + \sinh^2 \varphi_1 \, d\varphi_2^2 \right)$$

Peldan et al 1996

- Ansatz: $T(\tau), R(\tau), \Phi_1(\tau); \;\; \Phi_2 = p\sigma$
- Integral of motion: $K = T' f(R) = \text{const.}$
- EOMs:

$$\Phi_1'' + 2\frac{R'}{R}\Phi_1' + \frac{p^2}{2}\sinh 2\Phi_1 = 0$$

$$R'' - fR\left(\Phi_1'^2 - \sinh^2 \Phi_1\right) - \frac{f'}{2f}\left(R'^2 - f^2 T'^2\right) = 0$$

- 2D Hamiltonian: $H_{\text{eff}} = \frac{f(R)}{2} P_R^2 + \frac{1}{2R^2} P_{\Phi_1}^2 + \frac{K^2}{2f(R)} + p^2 R^2 \sinh^2 \Phi_1$

Integrable TBH: hyperbolic pendulum dynamics

- Canonical transformation:

$$S = s(\rho) P_{\Phi_1} \quad : \quad (R, \Phi_1) \quad \rightarrow \quad (\rho, \lambda)$$

$$H_{\text{eff}} = \frac{1}{2} P_\rho^2 + \frac{K^2}{2 s(\rho) f(\rho) (f'(\rho))^2} + \frac{s(\rho)}{2\rho^2} \left(P_\lambda^2 + p^2 \sinh^2 \lambda\right) = H_\rho + \frac{s(\rho)}{2\rho^2} H_\lambda$$

- Phase space foliated by tori at $H_\lambda = const.$
- Now in each subsystem it is possible to introduce action-angle variables if $s(\rho)/2\rho^2$ is a 1-1 mapping
- Don't know how to do this for general f. Works for:
  - $f = r^2 \pm 1 - 2m/r + q_x^2/r^2$ - extremal (all genuses)
  - $f = r^2 - 1 - 2m/r + q^2/r^2, \quad m \leq q/4$ - hyperbolic
  - higher genuses for special values of m, q

The one fixed point

- The only fixed point solution for hyperbolic, toric and higher genus horizons: $R = R_{0,} T = K / f\left(R_0\right), \quad \Phi_1 = 0$
- Rings a bell: need at least one stable and one unstable manifold for chaos

The one fixed point

- The only fixed point solution for hyperbolic, toric and higher genus horizons: $R=R_{0,} T=K/f(R_0), \quad \Phi_1=0$
- Rings a bell: need at least one stable and one unstable manifold for chaos
- Orbits:

 - scatter into infinity

 - make n orbits around the BH and then to infinity

 - make n orbits around the BH and then fall in

 - fixed point: balance at the extremal horizon or some distance R_0 around the horizon

Invariant plane and variational equations

- Invariant plane: $\left(P_R(\tau), P_{\Phi_1}=0, R(\tau), \Phi_1=0\right)$
- Variational equation in the tangent plane:

$$\delta\Phi''_1+2\left(\log R^0(\tau)\right)'\delta\Phi_1'+2p^2\delta\Phi_1=0$$

$$\delta R''+P_R^0(\tau)\left(1+f'\left(R^0(\tau)\right)\right)\delta R'+\partial_{R^0}\left(\frac{f'\left(R^0(\tau)\right)}{f\left(R^0(\tau)\right)^2}\right)\delta R=0$$

- Analytical solution in the invariant plane:

$$\sin\Phi(\tau)=sn\left(a(\tau-\tau_0),\frac{2p^2}{K^2},\frac{R_0 f_0^2}{f'_0}\right)$$

- For the extremal horizon we immediately establish integrability, variational equations reduce just to:

$$\delta\Phi''_1+p^2\delta\Phi_1=0,\quad \delta R''+P_R^0\delta R'=0$$

The Kovacic algorithm

- Automatic search for the center of the Galois group
- Practical recipe:

  - write down linearized perturbation equations in the plane tangent to an invariant manifold

  - check if the coefficients of $\delta X(\tau)$ can be expressed as rational functions of τ
- For the second step we typically need to transform the variable $u(\tau)$
- For (2b): $u(\tau)=F\left(\frac{\tau-\tau_0}{2gf(R_0(\tau))} \mid \frac{K}{2p^2 f(R_0(\tau))}\right)$
- In other cases I don't know -> `kovacicsol` open source tool for Maple (there are many others)

The outcome

- The hyperbolic black hole (2b) is always integrable for a closed winding string
- The spherical black hole (2b) is never integrable
- The toric and higher genus cases integrable for special values of BH mass and charge – for generic values the different invariant manifolds mix and spoil integrability

Numerical checks

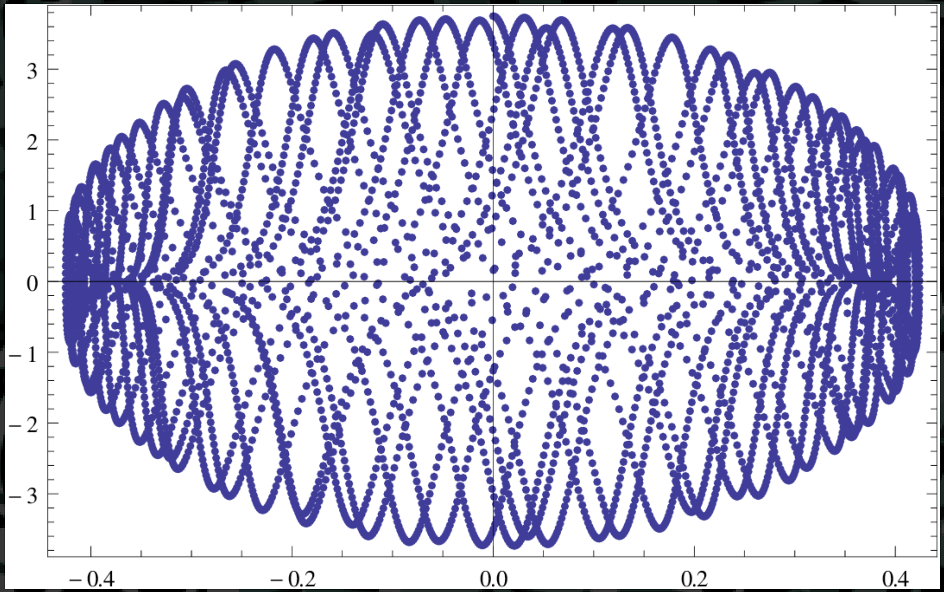
- Clearly no proof of integrability but can disprove it
- Careful: chaos -> nonintegrable but nonintegrable does not imply chaos – most noninterable string orbits are not chaotic!!!
- (1) Poincare surfaces of section (SOS) to visaulize the geometry of the phase space and KAM tori
- (2) Power spectrum – discrete -> integrable, continuous -> chaos; also bifurcations
- (3) Positive Lyapunov exponents (LE) -> chaos

KAM tori – hyperbolic horizon

- Direct visualization of KAM tori on Poincare surfaces of section (SOS)

$(R, P_R) \quad @ \quad \Theta = 0, P_\Theta > 0$

P_R



R

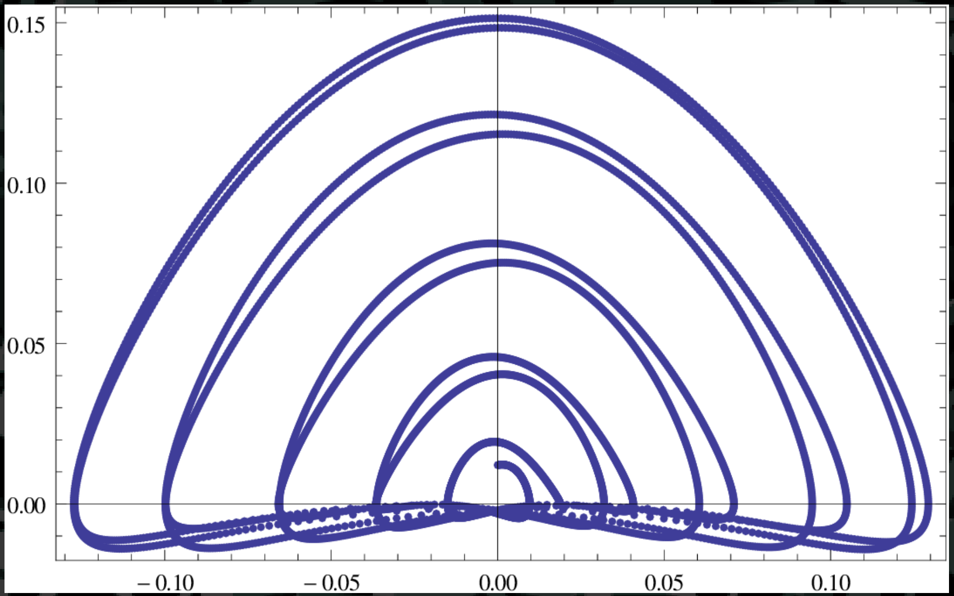
hyperbolic horizon $K = 23.84; \quad m = 1/2, q = 1/6$

KAM tori – toric horizon

- The orbits in real space do not make closed paths

(R, P_R) @ $\Theta=0, P_\Theta>0$

P_R



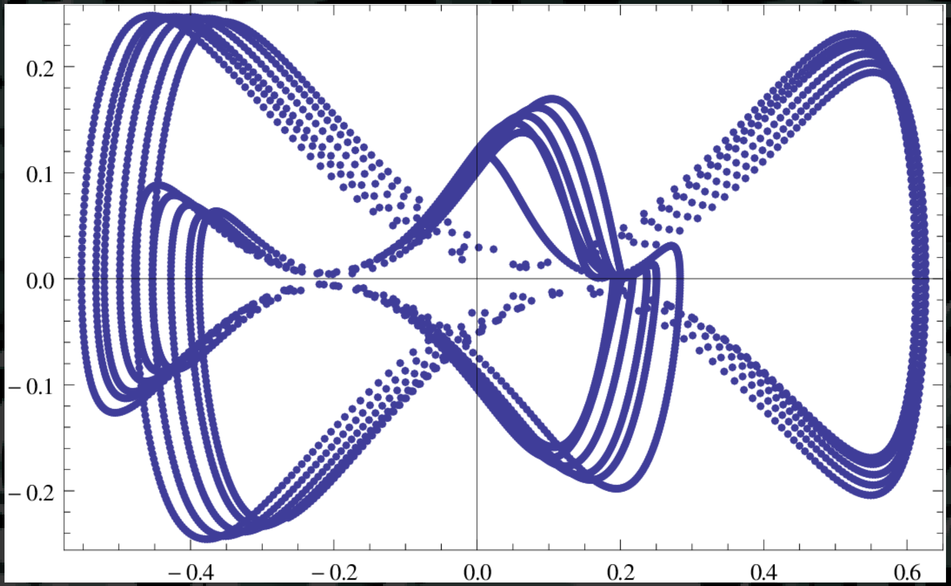
R

Toric horizon $K=0.32;\ m=1/2, q=1/6$

KAM tori – Brezel horizon

- Regular orbits for g=3

(R, P_R) @ $\Theta=0, P_\Theta>0$



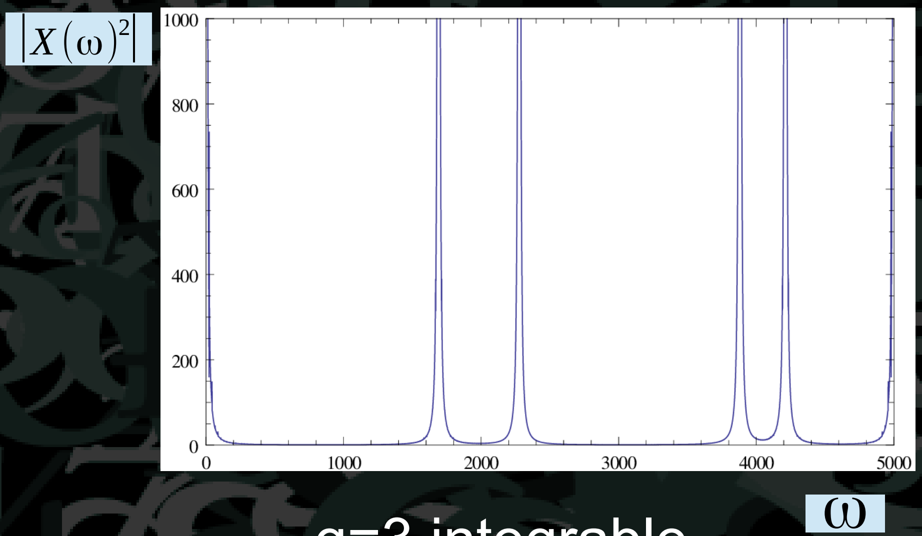
P_R

R

Spherical horizon $K=1.32;\quad m=1/2, q=1/6$

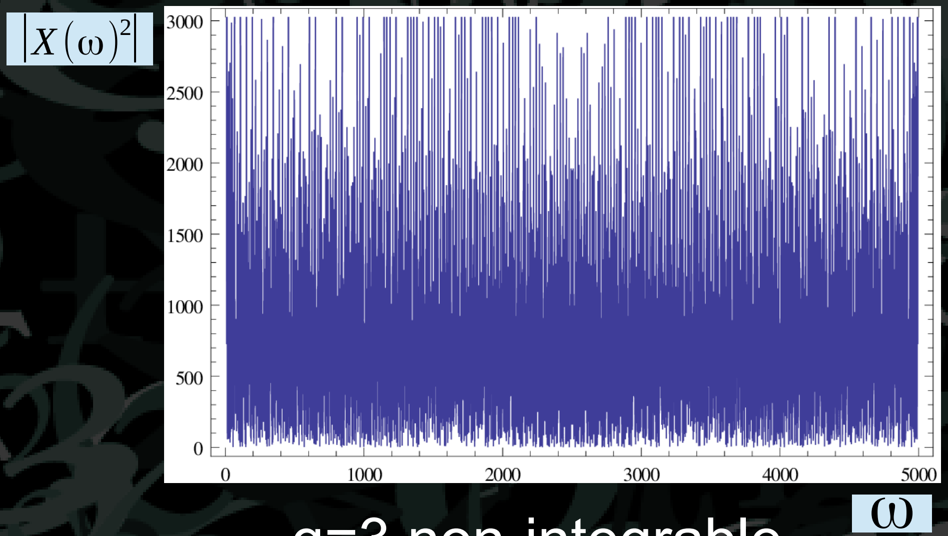
Power spectrum – Brezel horizon

- Quasi-periodic motion (not simply periodic – impossible for a string)



g=3 integrable

$m = 1/3$



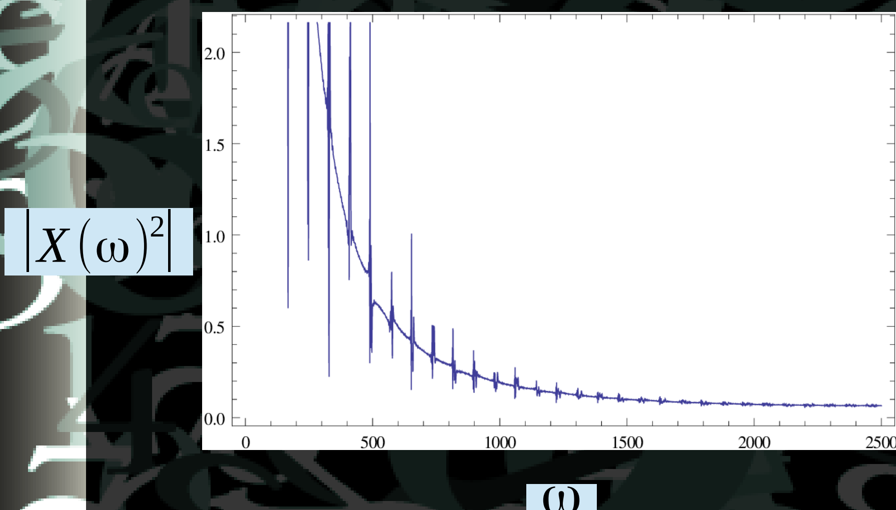
g=3 non-integrable

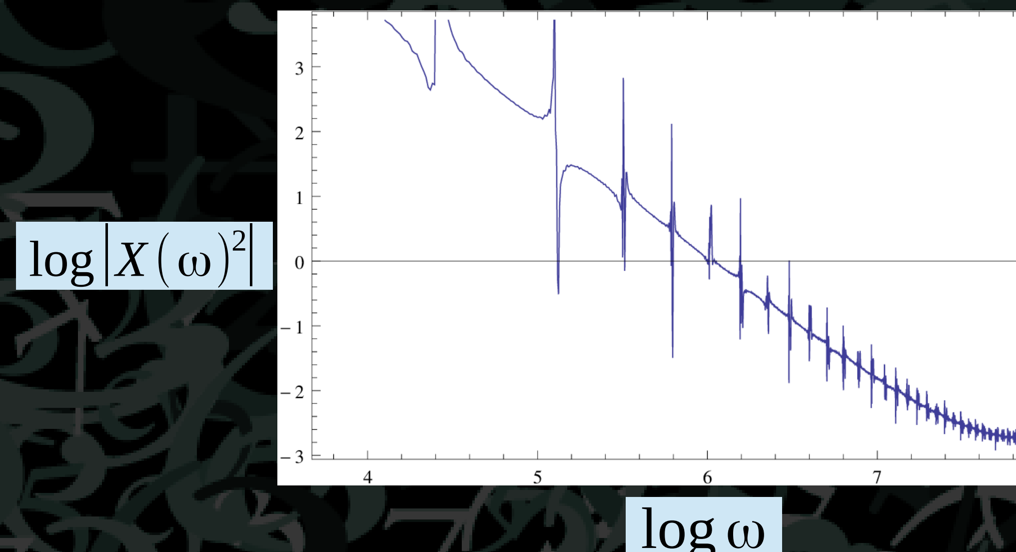
$m = 1/3 + 1/100$

- Quick jump to chaos unless very close to horizon (will come back to this)

Power spectrum – toric horizon

- Nonintegrable orbits exhibit universal Brownian spectrum: $X(\omega) = 1/\omega^2$ for toric horizon





- Apparently from the sum of many identical chaotic modes (integral of the white noise along the string)

Other cases

- (1a) Integrable for special mass & charge

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + \frac{L^4}{R_+^2} \cosh^2\left(\frac{R_+}{L^2} t\right)^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)$$

- Integral of motion: $K = \Theta_1{}' \cosh^2\left(R_+ t / L^2\right)$
- Weird – explicitly time-dependent integrable metric

Other cases

- (1a) Integrable for special mass & charge

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + \frac{L^4}{R_+^2} \cosh^2\left(\frac{R_+}{L^2} t\right)^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)$$

- Integral of motion: $K = \Theta_1{}' \cosh^2\left(R_+ t / L^2\right)$
- Weird – explicitly time-dependent integrable metric
- (2a) Non-integrable but has an extra integral of motion:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\varphi_1^2 + \frac{L^4}{R_+^2} \cosh^2\left(\frac{R_+}{L^2} t\right)^2 d\theta_1^2 \quad ; \quad K = \Phi_1{}' R^2$$

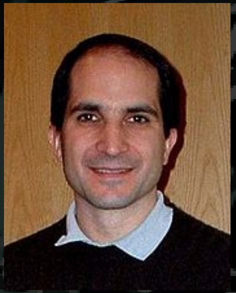
- (1b), (1ab), (2ab) – nope – the mixing of Φ -terms and T -terms spoils everything

Outline

- Integrability in nature and in string theory
- Topological black holes
- (Non)integrability – analytical and numerical
- Bound on chaos?

Conjecture on the bound on chaos

- Maldacena, Shenker & Stanford 2016: $\lambda \leq 2\pi T$ for QFTs
- This implies $\lambda \leq \kappa$ for BH horizons and $\lambda = 0$ for extremal BHs



J. Maldacena

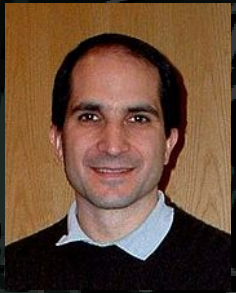
Conjecture on the bound on chaos

- Maldacena, Shenker & Stanford 2016: $\lambda \leq 2\pi T$ for QFTs
- This implies $\lambda \leq \kappa$ for BH horizons and $\lambda = 0$ for extremal BHs
- Idea of the proof:

(1) define LE from a correlation function: $C(t) = \langle [A(0), B(t)]^2 \rangle$

(2) show that $C(t + i\tau)$ is bounded by unity and analytic in the half-strip $0 \leq t, \beta/4 \geq \tau$

(3) apply the Schwarz-Pick theorem to obtain the bound



J. Maldacena

What could go wrong?

- (1) reasonable, (3) rigorous maths
- - correlation function might not factorize

Deep quantum effect

What could go wrong?

- (1) reasonable, (3) rigorous maths
- - correlation function might not factorize

Deep quantum effect

 - polynomial decay for weak chaos (no well-defined "collision time"~1/T)

What could go wrong?

- (1) reasonable, (3) rigorous maths
- - correlation function might not factorize

  - polynomial decay for weak chaos (no well-defined "collision time"~1/T)

Deep quantum effect

- Does it even work in curved spacetime?
- Many think yes. Sounds reasonable at least if there is a global timelike Killing
- In any case in asymptotically AdS should make sense through AdS/CFT

Sanity check – integrable systems have zero LE

- Systems (2b nonspherical) and (1a) have universal near-horizon variational equation: $\delta\Phi'' + 2n^2 \delta\Phi = 0$
- This means $\lambda = 0$ – notice the plus sign in front of the second term!

Higher winding modes in static metrics increase the bound

- No easy way to keep the string (or anything else) right at the horizon
- One approach: introduce external field to balance the horizon gravity (Hashimoto 2013) – but pair production? stability of the horizon?
- Expanding the variational equations near the horizon we get for stationary nonintegrable metrics (2a, 2ab):

$$\delta\Phi''-2\left(f'\left(R_h\right)\right)^2 n^2\delta\Phi=0$$

Naive LE: $\lambda_0\sim\kappa\times n$

- Higher winding numbers n violate the bound n times
- This wouldn't happen if there was a mass scale:

$$E_n\sim E_0+n^2$$

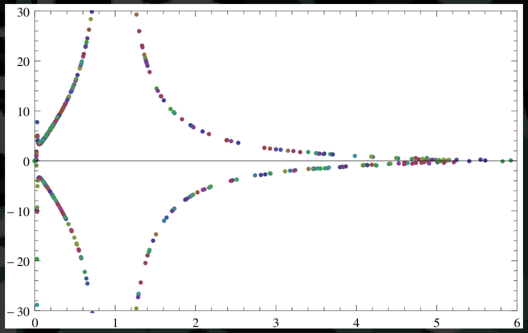
Non-static metrics do not obey the bound

- No universal near-horizon variational equation
- For non-integrable cases the EOMs remain complicated (no extra integrals of motion); generically

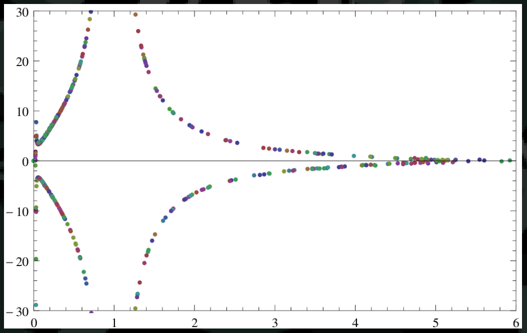
$$\delta\Phi'' + \frac{T^0}{f(R^0)}\delta\Phi' - 2f'(R_h)^2\delta\Phi = 0 \Rightarrow \lambda = \lim_{t\to\infty}\int_0^\infty d\tau' f'(R^0(\tau'))$$

- But this is perhaps expected – although staticity not assumed in the proof it plays a role in factorization of the OTOC

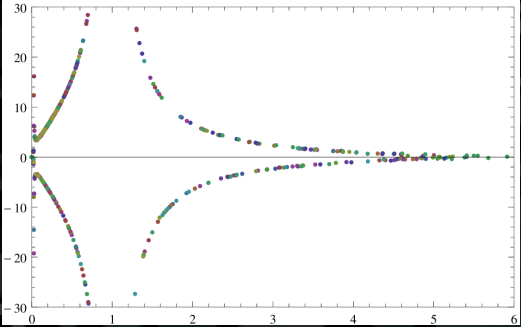
Regularity of T=0 at the horizon



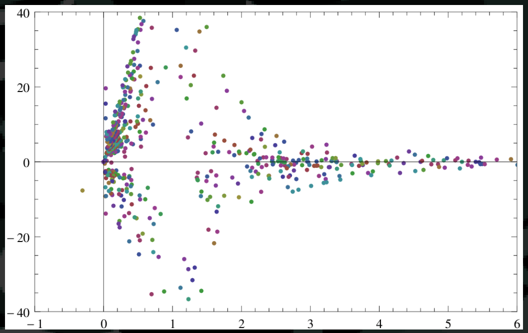
(1a) @ T=0
nonintegrable



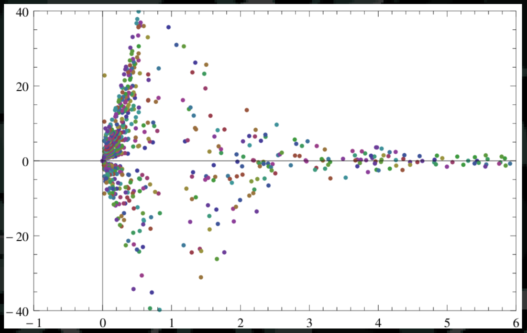
(1b) @ T=0
nonintegrable



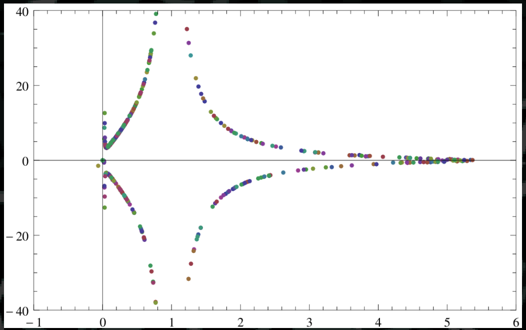
(2b) @ T=0
integrable



(1a) @ T=0.01
nonintegrable



(1b) @ T=0.01
nonintegrable



(2b) @ T=0.5
integrable

Some musings on the results...

- Understand TBHs. Cosmology? Or just AdS/CFT?
- Relation to AdS/CFT: look at open strings, these are connected to quarks in quark-gluon plasmas, tracer particles in hydrodynamics etc.
- Can we get $\lambda_0 \sim \kappa \times 2s$ for fields?